

SCHREIER THEOREM ON GROUPS WHICH SPLIT OVER FREE ABELIAN GROUPS

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ABSTRACT. Let G be either a free product with amalgamation $A *_C B$ or an HNN group $A *_C$, where C is isomorphic to a free abelian group of finite rank. Suppose that both A and B have no nontrivial, finitely generated, normal subgroups of infinite indices. We show that if G contains a finitely generated normal subgroup N which is neither contained in C nor free, then the index of N in G is finite. Further, as an application of this result, we show that the fundamental group of a torus sum of 3-manifolds M_1 and M_2 , the interiors of which admit hyperbolic structures, have no nontrivial, finitely generated, nonfree, normal subgroup of infinite index if each of M_1 and M_2 has at least one nontorus boundary.

INTRODUCTION

Some classes of groups are known to have no nontrivial, finitely generated, normal subgroup of infinite index. Let \mathcal{S} be the class of groups which have no nontrivial, finitely generated, normal subgroup of infinite index. Schreier showed in [4] that free groups are in \mathcal{S} . It is well-known that the fundamental groups of surfaces other than torus or Klein bottle are in \mathcal{S} , too. In the cases of torus and Klein bottle, the fundamental groups of them contain the infinite cyclic group as a nontrivial, finitely generated, normal subgroup of infinite index (see Proposition 15.23 in [6] for a proof). It is easy to see that both free groups and the fundamental groups of surfaces are either free products with amalgamation or HNN groups. According to Theorem 3.11 in [5], any free product with amalgamation $G = A *_C B$ with $C = \{1\}$ and $N \neq \{1\}$ also belongs to \mathcal{S} .

However, this result is not extended for groups of the form $A *_C B$ or $A *_C$, where both A and B are in \mathcal{S} and C is a free abelian group of finite rank. Let G be the fundamental group of the trefoil knot complement in S^3 . It is known that G has the presentation $\langle a, b; a^2 = b^3 \rangle$. Thus G is of the form $A *_C B$, where A, B and C are infinite cyclic groups. Let $N_1 = C$ and N_2 be the commutator subgroup of G . It can be easily shown that N_1 is normal in G and G/N_1 is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$. On the other hand, it can be easily observed that N_2 is the free group

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of rank 2 generated by $aba^{-1}b-1$ and $ab^2a^{-1}b^{-2}$. Since G/N_2 is isomorphic to the first homology group of the trefoil knot complement in S^3 , which is isomorphic to the infinite cyclic group, it follows that each of N_1 and N_2 is a finitely generated normal subgroup of infinite index. Hence G is not in \mathcal{S} .

Let G be a group of the form $A *_C B$ or $A *_C$, where $A, B \in \mathcal{S}$ and C is a free abelian group of finite rank, and let N be a nontrivial finitely generated normal subgroup of G . In this paper, we would like to give an answer to how N behaves when N fails to be of finite index in G . If N is not of finite index in G , either N is contained in C like N_1 in the above example or N intersects both A and B trivially like N_2 in the above example. Note that $N_2 \cap A = \{1\}$ and $N_2 \cap B = \{1\}$, as G/N_2 has no torsion element. In the latter case, N turns out to be free. The result we obtain is the following theorem.

Theorem 2.1. *Let $G = A *_C B$ or $G = A *_C$, where C is a free abelian group of finite rank. Suppose that both A and B contain no finitely generated nontrivial normal subgroup of infinite index. If N is a finitely generated normal subgroup of G with N not contained in C , then the index $|G : N|$ is finite or N is free.*

Ahlfors finiteness theorem implies that the fundamental group of a compact 3-manifold, the interior of which admits a hyperbolic structure, is in the class \mathcal{S} (see Theorem 3.1) if the manifold has at least one nontorus boundary. We derive Theorem 3.2 as an application of Theorem 2.1.

In §1 of this paper, we give a few technical lemmas on graphs of groups. In §2, we prove the main theorem, Theorem 2.1. In §3, we discuss an application of Theorem 2.1 to the fundamental group of a torus sum of two compact manifolds, each of which has at least one nontorus boundary component and admits a hyperbolic structure in its interior.

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1. GRAPH OF GROUPS

A graph of groups (Γ, G) is defined to be a connected graph Γ together with (a) groups G_v and G_e corresponding to each vertex v of Γ and edge e of Γ , and (b) monomorphisms $\phi_0 : G_e \rightarrow G_v$ and $\phi_1 : G_e \rightarrow G_w$ for each edge e of Γ , where v and w are the vertices of e .

Similarly, a graph of spaces (Γ, X) consists of a connected graph Γ together with (a) CW-complexes X_v and X_e corresponding to each vertex v and edge e of Γ , and (b) continuous maps $f_0 : X_e \rightarrow X_v$ and $f_1 : X_e \rightarrow X_w$ for each edge e of Γ , where v and w are the vertices of e . Given a graph of spaces, we can define a total space X_Γ as the quotient of

$$\bigcup \{X_v \mid v \text{ vertex of } \Gamma\} \cup \bigcup \{X_e \times I \mid e \text{ edge of } \Gamma\}$$

by the identifications

$$\begin{aligned} X_e \times 0 &\rightarrow X_v & \text{by } & (x, 0) \mapsto f_0(x), \\ X_e \times 1 &\rightarrow X_w & \text{by } & (x, 1) \mapsto f_1(x). \end{aligned}$$

If (Γ, X) is a graph of connected based spaces, then by taking fundamental groups, we obtain a graph of groups (Γ, G) with the same underlying abstract graph Γ . The fundamental group of the graph of groups (Γ, G) is defined to be the

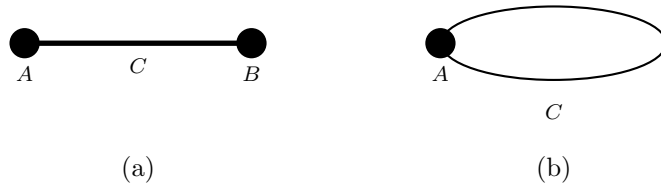


FIGURE 1. Free product with amalgamation and HNN group

fundamental group of the total space X_Γ . We denote by G_Γ a graph of groups with fundamental group G and underlying graph Γ .

For example, the free product with amalgamation $A *_C B$ is the fundamental group of the graph of groups in (a), Figure 1, where we have labelled the vertices and edge with the associated groups and $\phi_0 : C \rightarrow A$ and $\phi_1 : C \rightarrow B$ are the inclusions. In particular, if $G = A *_C B$ with $A \neq C \neq B$, we say that G has nontrivial amalgamation.

The HNN group $A *_C$ is obtained by adjoining an element t to A subject to the relations $t^{-1}at = \phi(a)$ for all $a \in A$, where C is a subgroup of a group G and $\phi : C \rightarrow A$ is a monomorphism. This group $A *_C$ can be considered as the fundamental group of the graph of groups of type (b) in Figure 1 when $\phi_0 : C \rightarrow A$ is the inclusion map and $\phi_1 : C \rightarrow A$ is the monomorphism ϕ . For example, when $A = C = \mathbb{Z}$, $A *_C$ is the fundamental group of either torus (if $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ is the identity) or Klein bottle (if $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ is the isomorphism defined by $\phi(x) = -x$).

A group G is said to split over a subgroup C if either $G = A *_C B$ with $A \neq C \neq B$ or $G = A *_C$. If G splits over some subgroup, we say that G is splittable. For example, \mathbb{Z} is splittable as $\mathbb{Z} = \{1\} *_{{\{1\}}}$.

Let H be a subgroup of G . Given a graph G_Γ of groups, there is a graph of spaces X_Γ corresponding to G_Γ . Consider the covering space Y_{Γ_1} of X_Γ whose vertex spaces and edge spaces are covering spaces of X_e and X_v for all edges e and vertices v of Γ with $\pi_1(X_{\Gamma_1}) = H$. Using covering space theory, one can derive the following lemma (see [5] for a proof).

Lemma 1.1. *If G is the fundamental group of a graph G_Γ of groups and $H < G$, then H is the fundamental group of a graph of groups, where each vertex group is the intersection of H and a conjugate of a vertex group of G_Γ and each edge group is the intersection of H and a conjugate of an edge group of G_Γ .*

For convenience, we denote a conjugate $g^{-1}Ag$ of A by A^g .

Let H_Γ be a graph of groups whose fundamental group H is finitely generated. If T is a maximal tree in Γ , $\Gamma \setminus T$ consists of finitely many edges as H is finitely generated. Let h_1, \dots, h_n be generators of H . Each $h_i, i = 1, \dots, n$, is a product of a finite number of elements either belonging to vertex groups or corresponding to the edges in $\Gamma \setminus T$. Thus h_1, \dots, h_n are contained in a finite subgraph of groups of H_Γ . It follows that there is a finite subgraph of groups of H_Γ which carries the whole fundamental group. Hence we obtain the following lemma.

Lemma 1.2. *Let H_Γ be a graph of groups whose fundamental group H is finitely generated. Then there is a finite subgraph of groups of H_Γ whose fundamental group is H .*

We can define the distance between two vertices in a graph and the diameter of a graph.

Definition 1.1. For any two vertices P and Q of a connected graph Γ , we define the distance $d(P, Q)$ between P and Q to be

$$\min\{n \mid P \text{ and } Q \text{ are connected by } n \text{ consecutive edges}\}.$$

We say Γ is of finite diameter if

$$\sup\{d(P, Q) \mid P \text{ and } Q \text{ are vertices of } \Gamma\}$$

is finite.

Suppose that G splits over a subgroup C . For a graph of groups G_Γ of type (a) or (b) in Figure 1, if the corresponding graph of groups to a normal subgroup of G is of finite diameter, then it is in fact a finite graph of groups. In particular, we do not need the subgroup to be normal when G is a free product with amalgamation.

Recall that a vertex v in a graph is of valence 1 if v has only one adjacent edge.

Lemma 1.3. *Let $G = A *_C B$ with $A \neq C \neq B$. Suppose G_Γ is a graph of groups of type (a) or (b) in Figure 1. If H is a subgroup of G and its corresponding graph of groups H_{Γ_1} has finite diameter, then it is a finite graph of groups.*

Proof. Let X_Γ and \tilde{X}_{Γ_1} be the graph of spaces corresponding to G_Γ and H_{Γ_1} , respectively, so that \tilde{X}_{Γ_1} is a covering space of X_Γ . First, we will show that there are only finitely many vertices of valence 1 in H_{Γ_1} . Let \tilde{v} be a vertex of valence 1 in H_{Γ_1} such that $H_{\tilde{v}} = H_{\tilde{e}}$ for the only edge \tilde{e} having \tilde{v} as its vertex. Let $\tilde{X}_{\tilde{v}}$ and $\tilde{X}_{\tilde{e}}$ be the vertex space and edge space corresponding to $H_{\tilde{v}}$ and $H_{\tilde{e}}$, respectively. The covering transformations fixing $\tilde{X}_{\tilde{v}}$ also fixes $\tilde{X}_{\tilde{e}}$ as \tilde{v} is a vertex of valence 1. So, in X_Γ , the fundamental group of one vertex space is the same as the fundamental group of the edge space. This implies that $A = C$ or $B = C$. This contradicts the hypothesis. Thus every vertex of valence 1 in H_{Γ_1} yields a nontrivial amalgamation. Since H is finitely generated, there are only finitely many vertices of valence 1.

Suppose Γ_1 has an infinite number of vertices of valence greater than one. Γ_1 cannot have infinitely many loops as H is finitely generated. By deleting a finite number of edges, we obtain a connected subtree T of Γ_1 with infinitely many vertices, all but a finite number of which are of valence greater than 1. Since all but a finite number of the vertices in T has valence greater than 1 and T is a tree, we can construct an infinite number of consecutive edges in T . This implies that T has infinite diameter, and so Γ_1 has infinite diameter. This is a contradiction to the hypothesis. Hence there are only finitely many vertices of valence greater than 1.

Now we know that there are only finitely many vertices in H_{Γ_1} . Note that a graph having finitely many vertices and infinitely many edges must have infinitely many loops. This is impossible as H is finitely generated. Therefore, H_{Γ_1} is a finite graph. □

Lemma 1.4. *Let $G = A *_C$ and let G_Γ be the corresponding graph of group of type (b) in Figure 1. If N is a normal subgroup of G and its corresponding graph of groups N_{Γ_1} has finite diameter, then it is a finite graph of groups.*

Proof. Let X_Γ and \tilde{X}_{Γ_1} be the graph of spaces corresponding to G_Γ and N_{Γ_1} , respectively, so that \tilde{X}_{Γ_1} is a covering space of X_Γ . We first consider the case where Γ_1 has a vertex of valence 1. If there is a vertex \tilde{v} of valence 1 in N_{Γ_1} , then every vertex in N_{Γ_1} is of valence 1 as the covering is regular. Thus the only possible type

for Γ_1 is the graph with two vertices and one edge connecting them. Hence N_{Γ_1} is a finite graph of groups.

Now suppose all of the vertices in Γ_1 are of valence greater than 1. By the same argument as in the proof of Lemma 1.3, we obtain a contradiction. Hence there are only finitely many vertices of valence 1.

Since Γ_1 has a finite number of vertices and H is finitely generated, Γ_1 is a finite graph. □

For a finite graph of groups whose fundamental group is finitely generated, all vertex groups are finitely generated under the condition that all edge groups are finitely generated.

Lemma 1.5. *Let H be a finitely generated group. If H_Γ is a finite graph of groups whose fundamental group is H , and if all edge groups of H_Γ are finitely generated, then all vertex groups of H_Γ are also finitely generated.*

Proof. Let H_v be an arbitrary vertex group of H_Γ . Since Γ is finite, there are only finitely many edges which have v as their common vertex. Let e_1, \dots, e_k be such edges. Let $V_1 = \langle H_{e_1}, \dots, H_{e_k} \rangle$. Assume H_v is not finitely generated. Then there is an infinite sequence of finitely generated subgroups of H_v such that

$$V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \dots \quad \text{and} \quad \bigcup_{i=1}^{\infty} V_i = H_v.$$

Since the H_{e_j} 's are subgroups of V_i 's, we can construct a graph X_i of groups from H_Γ by replacing H_v with V_i . Set W_i to be the fundamental group of X_i . Then $W_i \cap H_v = V_i$. Hence we have an infinite sequence $\{W_i\}$ of subgroups of H such that

$$W_1 \subsetneq W_2 \subsetneq W_3 \subsetneq \dots$$

and $\bigcup_{i=1}^{\infty} W_i = H$. It follows that H is not finitely generated, which is a contradiction. Therefore, H_v is finitely generated. □

2. GROUPS SPLITTING OVER FREE ABELIAN GROUPS OF FINITE RANK

Let G be a group which splits over a group C . Suppose G has a finitely generated normal subgroup N . Consider the graph of groups G_Γ corresponding to G of type either (a) or (b) in Figure 1. There is a corresponding graph of groups N_{Γ_1} to N , so that the corresponding graph of spaces Y_{Γ_1} to N_{Γ_1} is a covering space of the graph of spaces X_Γ corresponding to G_Γ . If Γ_1 is of infinite diameter, then it turns out that N is contained in C .

Lemma 2.1. *Let $G = A *_C B$ or $G = A *_C$ with a finitely generated normal subgroup N . If the corresponding graph of groups N_{Γ_1} with vertex groups $\{N \cap A^g : g \in G\} \cup \{N \cap B^g : g \in G\}$ and edge groups $\{N \cap C^g : g \in G\}$ is of infinite diameter, then N is contained in C .*

Proof. If Γ_1 is of infinite diameter, then Γ_1 is a tree. In fact, if Γ_1 were not a tree, there would be an edge which is not in a maximal tree of Γ_1 . Such an edge would be a part of a loop L . Since the covering is regular and Γ_1 is of infinite diameter, there is a covering transformation g_1 such that $g_1L \cap L = \emptyset$. By the same reason, there are an infinite number of covering transformations $g_n, n = 1, 2, \dots$, such that L, g_1L, g_2L, \dots are mutually disjoint. Thus there would be an infinite number of

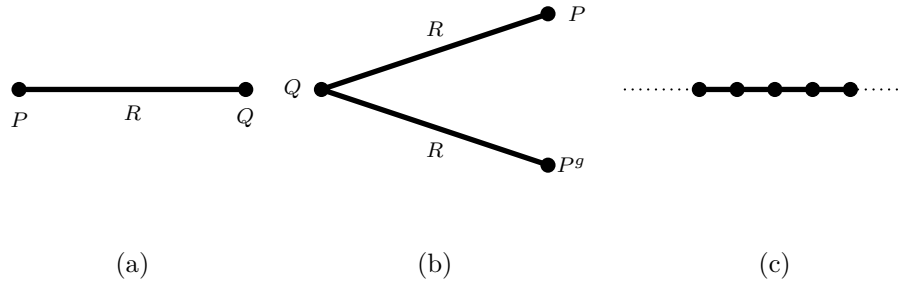


FIGURE 2.

loops in Γ_1 . This is a contradiction to the fact that N is finitely generated. It follows that Γ_1 is a tree.

Since N is finitely generated, there is a finite subgraph of groups N_{Γ_2} of N_{Γ_1} whose fundamental group is N . It follows that there is an infinite subtree of groups T whose fundamental group is a vertex group. Also, in T , there is a tree of groups consisting of two vertex groups and one edge group and with trivial amalgamation, i.e., one of the vertex groups is the same as the edge group. Let E be the graph of groups with two vertex groups P and Q , and one edge group R satisfying $Q = R$ as in (a), Figure 2. Since the covering is regular, every subgraph of groups with two vertex groups and one edge group is of the same type as E . We have two cases to consider.

In the case where $G = A *_C B$, P should be the same as R . If not, we have a tree of groups of type (a) in Figure 2 with $P \neq R$, which produces a tree of groups of type (b) in Figure 2 with $P \neq R \neq P^g$ and $Q = R$. It is impossible to have a tree of groups of such type in T , as a vertex group of T carries the whole fundamental group of T . Now, since $P = R = Q$, every vertex group in N_{Γ_1} is an edge group. Hence the fundamental group N of N_{Γ_1} is an edge group R , which is a subgroup of C . It follows that N is contained in C .

In the case where $G = A *_C$, Γ_1 should be a tree of type (c) in Figure 2 as Γ_1 is a tree and the covering is regular. If $P \neq R$, $N = \bigcup_{-\infty}^{\infty} N_i$, where each N_i is a vertex group and $N_i \subsetneq N_{i+1}$. Since N is finitely generated, it is impossible. Hence $P = R$, and so N is an edge group. It follows that N is contained in C . \square

Now we are in a position to prove the main theorem of this paper.

Theorem 2.1. *Let $G = A *_C B$ or $G = A *_C$, where C is a free abelian group of finite rank. Suppose that both A and B contain no finitely generated nontrivial normal subgroup of infinite index. If N is a finitely generated normal subgroup of G with N not contained in C , then the index $|G : N|$ is finite or N is free.*

Proof. We will prove the theorem in the case where $G = A *_C B$. The case for $G = A *_C$ is the same. Let G_{Γ} be the graph of groups with two vertex groups A and B , and one edge group C . There is a corresponding graph of groups N_{Γ_1} to N , so that the corresponding graph of spaces Y_{Γ_1} to N_{Γ_1} is a covering space of the graph of spaces X_{Γ} corresponding to G_{Γ} . Since N is not contained in C , Lemma 2.1 assures us that N_{Γ_1} is of finite diameter. By Lemma 1.3 and Lemma 1.4, N_{Γ_1} is a finite graph of groups. Since C is a free abelian group of finite rank, all the

edge groups of N_{Γ_1} are finitely generated. By Lemma 1.5, all the vertex groups are finitely generated.

If both $N \cap A$ and $N \cap B$ are trivial, all the vertex groups and edges groups of N_{Γ_1} are trivial. It follows that N is free.

Now suppose that $N \cap A \neq \{1\}$ or $N \cap B \neq \{1\}$. Without loss of generality, we may assume that $N \cap A \neq \{1\}$. Then the index $|A : N \cap A|$ is finite, as $N \cap A$ is a finitely generated nontrivial normal subgroup of A . It follows that the covering is finite, as the covering is regular. Hence the index $|G : N|$ is finite. \square

3. APPLICATION TO KLEINIAN GROUPS

Recall that a Kleinian group is a discrete subgroup of $PSL_2(\mathbb{C})$, which is the group of isometries of \mathbb{H}^3 , the 3-dimensional hyperbolic space. A hyperbolic 3-manifold is defined to be a manifold which is the quotient of \mathbb{H}^3 by a torsion free Kleinian group G acting as a covering group.

Thinking of \mathbb{H}^3 as the interior of the closed 3-ball B^3 , we may think of G as a group of conformal transformations of S_∞^2 , which is ∂B^3 , the sphere at infinity. This action and the action on H^3 fit together to form an action on the closed ball B^3 . The group G acts properly on $Int(B^3)$, but it does not act properly discontinuously on S_∞^2 . The limit set L_G of G is defined to be

$$\{x \in S_\infty^2 \mid g_n(y) \rightarrow x \text{ for some } y \in \mathbb{H}^3 \text{ and } g_n \in G\}.$$

The complement of L_G in S_∞^2 is called the domain of discontinuity of G and denoted by D_G .

The limit set of a Kleinian group contains either less than 3 elements or an infinite number of elements. A Kleinian group G is called nonelementary if L_G contains an infinite number of elements. It is known that if N is a nontrivial normal subgroup of a nonelementary Kleinian group G , then $L_G = L_N$.

A Kleinian group G is said to be of 2-nd kind if L_G is not the whole sphere S_∞^2 .

Let M be a compact 3-manifold which has at least one nontorus boundary component and hyperbolic structure in its interior. M is homeomorphic to a 3-manifold $(\mathbb{H}^3 \cup D_G)/G$ with the nonempty boundary D_G/G , where G is a nonelementary Kleinian group of 2-nd kind (see [2] or [3]). Ahlfors's finiteness theorem says that if G is a finitely generated discrete torsion-free Kleinian group, then D_G is a finite collection of finite area hyperbolic surfaces (see [1]). This implies the following theorem.

Theorem 3.1. *Let G be the fundamental group of a compact 3-manifold M which has a nontorus boundary component and hyperbolic structure in its interior. If N is a nontrivial, finitely generated, normal subgroup of G , then the index $|G : N|$ is finite.*

Proof. Since N is normal in G and G is a nonelementary Kleinian group, $L_N = L_G$. It follows that $D_G = D_N$. Since G is a Kleinian group of 2-nd kind, $D_G \neq \emptyset$. By the Ahlfors finiteness theorem, D_G/G and D_N/N are finite area hyperbolic surfaces. Hence the covering $D_N/N \rightarrow D_G/G$ must be finite. Therefore N is of finite index in G . \square

Let M be a torus sum of two compact 3-manifolds each of which has at least one nontorus boundary component and has hyperbolic structure in its interior. Then the fundamental group of M is a free product with amalgamation $A *_C B$, where

both A and B are nonelementary Kleinian groups of 2-nd kind and C is a torus group. Theorem 3.1 and Theorem 2.1 produce the following theorem.

Theorem 3.2. *Let G be the fundamental group of a torus sum of two compact 3-manifolds each of which has at least one nontorus boundary component and has hyperbolic structure in its interior. If N is a nontrivial, finitely generated, normal subgroup of G , then the index $|G : N|$ is finite or N is free.*

Proof. By Theorem 3.1, both A and B have no finitely generated normal subgroup of infinite index. If N were in C , then N would be a finitely generated normal subgroup of infinite index of A , which is impossible. Hence N is not contained in C . By Theorem 2.1, $|G : N|$ is finite or N is free. \square

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