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OBLIQUE MULTIWAVELETS IN HILBERT SPACES

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ABSTRACT. In this paper, we elucidate the relationship between two consecutive levels of a multiresolution in the general setting of a Hilbert space. We first prove a result on an extendability problem and then derive, as a consequence, characterizations of oblique multiwavelets in a Hilbert space.

1. INTRODUCTION

This is a sequel to our earlier papers [4, 7, 5, 10, 8]. Here, we again consider wavelet-type problems in a general Hilbert space setting, rather than in the usual concrete case of $L^2(\mathbf{R}^d)$, the Hilbert space of square integrable complex-valued functions on $\mathbf{R}^d$.

It is well known that one of the major issues in the study of a multiresolution $\{V_n\}$ of $L^2(\mathbf{R}^d)$ (see e.g. [3, 6]) is the relationship between any two consecutive multiresolution spaces V_n and V_{n+1} (between only V_0 and V_1 in the *stationary* case). This paper elucidates this relationship in the setting of a Hilbert space.

In [1], wavelet-type objects called *oblique multiwavelets* in $L^2(\mathbf{R})$ are introduced. These are generalizations of biorthogonal multiwavelets and it is noted in [1] that they have more flexible properties. Characterizations of oblique multiwavelets in $L^2(\mathbf{R})$ are obtained in [1] and [2], using the machinery of the Fourier transform on $L^2(\mathbf{R})$. This paper extends these characterizations to the general setting of oblique multiwavelets in a Hilbert space H, where the Fourier transform is no longer available and the roles of the translation operator and the dilation operator on $L^2(\mathbf{R})$ are replaced by certain unitary operators on H. Indeed, we first prove a result on a related extendability problem and then obtain, as a consequence, the above results on oblique multiwavelets in a Hilbert space.

Throughout this paper, H denotes a complex Hilbert space. A sequence $\{v_n\}$ in H is a *Riesz basis* for its closed linear span $\overline{span}\{v_n\}$ if there exist positive constants A and B such that

$$A\sum |a_n|^2 \le \| \sum a_n v_n\|^2 \le B\sum |a_n|^2, \quad \forall \{a_n\} \in \ell^2. \tag{1.1}$$

Two sequences $\{v_n\}$ and $\{\tilde{v}_n\}$ in H are *biorthogonal* if

$$\langle v_n, \tilde{v}_m\rangle = \delta_{n,m} \quad \forall n, m, \tag{1.2}$$

where $\langle x, y\rangle$ denotes the inner product of two vectors x and y in H. (See [11].)

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Let $U = (U_1, ..., U_d)$ be an ordered d-tuple of distinct unitary operators on a Hilbert space H such that $U_k U_j = U_j U_k$, $k, j = 1, \ldots, d$. We shall use the multi-index notation $U^m = U_1^{m_1} \cdots U_d^{m_d}$ for $m = (m_1, ..., m_d) \in \mathbf{Z}^d$, with the convention that U_j^0 is the identity operator on H, $j = 1, ..., d$. We also assume that U^m is the identity operator only if $m = 0$.

For a subset S of H, let $\langle S \rangle$ denote the closed linear span of S, and

$$U^{\mathbf{Z}^d}(S) := \{U^n s : n \in \mathbf{Z}^d, s \in S\}.$$

If $V = \{v_1, ..., v_r\}$ and $W = \{w_1, ..., w_p\}$ are finite subsets of H such that

$$\{\langle v_k, U^n w_j \rangle\}_{n \in \mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d), \quad k = 1, \ldots, r,\ j = 1, \ldots, p, \tag{1.3}$$

then the function $\Phi_{V,W}$ defined almost everywhere on $\mathbf{R}^d$ by

$$\Phi_{V,W}(u) := \Big(\sum_{n \in \mathbf{Z}^d} \langle v_k, U^n w_j \rangle e^{in \cdot u} \Big)_{1 \le k \le r,\, 1 \le j \le p} \tag{1.4}$$

is in $L^2(\mathbf{T}^d)^{r \times p}$, the space of all $r \times p$ matrices with entries in $L^2(\mathbf{T}^d)$ (often identified with $L^2(\mathbf{T}^d, \mathbf{C}^{r \times p})$, where $\mathbf{C}^{r \times p}$ is the space of all $r \times p$ complex matrices equipped with the operator norm, or depending on the context, with the Frobenius norm). For simplicity, we write $L^2(\mathbf{T}^d)^p$ for $L^2(\mathbf{T}^d)^{1 \times p}$.

If V and W are closed linear subspaces of H such that $V \cap W = \{0\}$ and the vector sum $V_1 = V + W$ is closed, then we write $V_1 = V \oplus W$ and call this a *direct sum*. In this case, we define the maps $P_{V//W}$ and $P_{W//V}$ from V_1 to V_1 by

$$P_{V//W}(v + w) = v, \quad P_{W//V}(v + w) = w, \quad v \in V, w \in W\,, \tag{1.5}$$

and call $P_{V//W}$ the (oblique) projection of V_1 on V along W and $P_{W//V}$ the (oblique) projection of V_1 on W along V. For the special case when $W = V_1 \cap V^\perp$, the orthogonal complement of V in V_1, we write $V_1 = V \oplus^\perp W$ for the *orthogonal* direct sum.

We now give a summary of the contents of this paper. In section 2, we obtain necessary and sufficient conditions for the existence of solutions to an extendability problem in a Hilbert space that is related to multiresolution. Section 3 summarizes some useful results on the connection between $L^2(\mathbf{T}^d)$ and dilation matrices. The final section elucidates the relationship between two consecutive levels of a multiresolution-type structure and derives characterizations of oblique multiwavelets in a Hilbert space.

2. An extendability problem

Throughout this section, let $Y = \{y_1, ..., y_s\}$ and $\widetilde{Y} = \{\tilde{y}_1, ..., \tilde{y}_s\}$ be finite subsets of H such that $U^{\mathbf{Z}^d}(Y)$ and $U^{\mathbf{Z}^d}(\widetilde{Y})$ are biorthogonal Riesz bases for $V_1 = \langle U^{\mathbf{Z}^d}(Y) \rangle = \langle U^{\mathbf{Z}^d}(\widetilde{Y}) \rangle$. Moreover, let $X = \{x_1, ..., x_r\}$ be a finite subset of H, where $r < s$, let $U^{\mathbf{Z}^d}(X)$ be a Riesz basis for its closed linear span V_0 and let

$$V_0 \subset V_1. \tag{2.1}$$

In this case, we have the biorthogonal expansions

$$f = \sum_{j=1}^{s} \sum_{n \in \mathbf{Z}^d} \langle f, U^n \tilde{y}_j \rangle U^n y_j = \sum_{j=1}^{s} \sum_{n \in \mathbf{Z}^d} \langle f, U^n y_j \rangle U^n \tilde{y}_j, \quad f \in V_1\,, \tag{2.2}$$

$$\Phi_{Y,\widetilde{Y}} = \Phi_{\widetilde{Y},Y} = I_s \ \ a.e., \tag{2.3}$$

where I_s is the $s \times s$ identity matrix, and the entries of the matrix functions $\Phi_{V,Y}$ and $\Phi_{V,\widetilde{Y}}$ are L^2-functions for any finite subset V of H.

Theorem 2.1. *Let $\Gamma = \{z_1, ..., z_{s-r}\}$ be a subset of $V_1 \setminus V_0$ and let $S = X \cup \Gamma$. The following conditions are equivalent:*

(i) $U^{\mathbf{Z}^d}(S)$ *is a Riesz basis for* V_1.

(ii) $U^{\mathbf{Z}^d}(\Gamma)$ *is a Riesz basis for* $W_0 := \langle U^{\mathbf{Z}^d}(\Gamma)\rangle$ *and* $V_0 \oplus W_0 = V_1$.

(iii) *There exist positive constants C_1 and C_2 such that*

$$C_1 \leq \Phi_{S,\widetilde{Y}}(u)\, \Phi_{S,\widetilde{Y}}(u)^* \leq C_2 \quad a.e.$$

(iv) *The matrices $\Phi_{S,\widetilde{Y}}(u)$ are invertible for almost all u, and the functions $u \longrightarrow \|\Phi_{S,\widetilde{Y}}(u)\|$ and $u \longrightarrow \|\Phi_{S,\widetilde{Y}}(u)^{-1}\|$ are essentially bounded.*

(v) *The operator $R_0 : L^2(\mathbf{T}^d)^r \oplus L^2(\mathbf{T}^d)^{s-r} \longrightarrow L^2(\mathbf{T}^d)^s$ defined by*

$$\begin{aligned} R_0(A,B)(u) &= (A(u) \ \ B(u)) \begin{pmatrix} \Phi_{X,\widetilde{Y}}(u) \\ \Phi_{\Gamma,\widetilde{Y}}(u) \end{pmatrix} \\ &= A(u)\, \Phi_{X,\widetilde{Y}}(u) + B(u)\, \Phi_{\Gamma,\widetilde{Y}}(u) \end{aligned} \tag{2.4}$$

is bounded and invertible.

Proof. (i)$\Longrightarrow$(iii) follows from [10, Proposition 3.4].

(iii)$\Longrightarrow$(ii): Again by [10, Proposition 3.4], $U^{\mathbf{Z}^d}(S)$ is a Riesz basis for $\langle U^{\mathbf{Z}^d}(S)\rangle$, which is a subset of V_1. In particular, the set S is linearly independent, $U^{\mathbf{Z}^d}(\Gamma)$ is a Riesz basis for $W_0 := \langle U^{\mathbf{Z}^d}(\Gamma)\rangle$ and

$$V_0 \oplus W_0 = \langle U^{\mathbf{Z}^d}(S)\rangle \subset V_1 = \langle U^{\mathbf{Z}^d}(Y)\rangle.$$

Since $\#(S) = s = \#(Y)$, by [5, Theorem 2.4], $\langle U^{\mathbf{Z}^d}(S)\rangle = V_1$.

By [10, Theorem 2.1], we have (ii)$\Longrightarrow$(i). Finally, (iii)$\Longleftrightarrow$(iv) follows from standard arguments in operator theory, and (iv)$\Longleftrightarrow$(v) follows from [9, pp. 351-352]. □

It is shown in [5, Theorem 2.5] ([7, Corollary 3.4] for the case $d = 1$) that there exist $z_1, ..., z_{s-r}$ in $V_1 \cap V_0^{\perp}$ such that condition (ii) in Theorem 2.1 holds. In the more general setting of a pair of biorthogonal multiresolutions, [10, Theorem 3.6] shows that again there exist $z_1, ..., z_{s-r}$ satisfying Theorem 2.1 (ii) and some other conditions.

Proposition 2.2. *Suppose that the conditions in Theorem 2.1 hold. Let $\widetilde{S}$ be a subset of V_1 such that $U^{\mathbf{Z}^d}(\widetilde{S})$ is a Riesz basis for V_1 biorthogonal to $U^{\mathbf{Z}^d}(S)$. Write $\widetilde{S} = X' \cup \Gamma'$ such that $\#(X') = r, \#(\Gamma') = s - r$, $U^{\mathbf{Z}^d}(X')$ biorthogonal to $U^{\mathbf{Z}^d}(X)$, $U^{\mathbf{Z}^d}(\Gamma')$ biorthogonal to $U^{\mathbf{Z}^d}(\Gamma)$, $U^{\mathbf{Z}^d}(X') \perp U^{\mathbf{Z}^d}(\Gamma)$ and $U^{\mathbf{Z}^d}(\Gamma') \perp U^{\mathbf{Z}^d}(X)$. Then $R_0^{-1} : L^2(\mathbf{T}^d)^s \longrightarrow L^2(\mathbf{T}^d)^r \oplus L^2(\mathbf{T}^d)^{s-r}$ is given by*

$$R_0^{-1}(C)(u) = (C(u)\Phi_{X',Y}(u)^*, \ \ C(u)\Phi_{\Gamma',Y}(u)^*). \tag{2.5}$$

Proof. By [5, Theorem 2.4], $\Phi_{S,\widetilde{Y}}(u)^{-1} = \Phi_{\widetilde{S},Y}(u)^* = (\Phi_{X',Y}(u)^* \ \ \Phi_{\Gamma',Y}(u)^*)$. By Theorem 2.1, $R_0^{-1}(C)(u) = C(u)\Phi_{S,\widetilde{Y}}(u)^{-1} = (C(u)\Phi_{X',Y}(u)^* \ \ C(u)\Phi_{\Gamma',Y}(u)^*)$. □

Remark 2.3. (i) Under the assumptions of Proposition 2.2, let $V_0' = \langle U^{\mathbf{Z}^d}(X')\rangle$ and $W_0' = \langle U^{\mathbf{Z}^d}(\Gamma')\rangle$. Then we arrive at the biorthogonal setting

$$V_0 \oplus W_0 = V_1, \quad V_0' \oplus W_0' = V_1, \quad V_0 \perp W_0', \quad V_0' \perp W_0.$$

Note that V_0' may not necessarily equal V_0 and W_0' may not necessarily equal W_0.

(ii) If moreover $U^{\mathbf{Z}^d}(X) \perp U^{\mathbf{Z}^d}(\Gamma)$, then $V_0' = V_0$ and $W_0' = W_0$, and we arrive at the semiorthogonal setting $V_0 \oplus^{\perp} W_0 = V_1$.

Theorem 2.4. *Suppose that the conditions in Theorem 2.1 hold. Let*

$$\begin{aligned} A_j(u) &= \sum_{n\in\mathbf{Z}^d} a_j(n)e^{in\cdot u}, \;\; a_j \in \ell^2(\mathbf{Z}^d), \;\; j=1,...,r, \\ B_k(u) &= \sum_{n\in\mathbf{Z}^d} b_k(n)e^{in\cdot u}, \;\; b_k \in \ell^2(\mathbf{Z}^d), \;\; k=1,...,s-r, \\ C_\ell(u) &= \sum_{n\in\mathbf{Z}^d} c_\ell(n)e^{in\cdot u}, \;\; c_\ell \in \ell^2(\mathbf{Z}^d), \;\; \ell=1,...,s, \end{aligned}$$

and

$$A = (A_1 \;\dots\; A_r), \;\; B = (B_1 \;\dots\; B_{s-r}), \;\; C = (C_1 \;\dots\; C_s).$$

The following conditions are equivalent:

(i) $$\sum_{\ell=1}^{s}\sum_{n\in\mathbf{Z}^d} c_\ell(n)U^n y_\ell = \sum_{j=1}^{r}\sum_{n\in\mathbf{Z}^d} a_j(n)U^n x_j + \sum_{k=1}^{s-r}\sum_{n\in\mathbf{Z}^d} b_k(n)U^n z_k.$$

(ii) (*Reconstruction algorithm*)

$$\begin{aligned} C(u) &= R_0(A,B)(u) \\ &= A(u)\Phi_{X,\widetilde{Y}}(u) + B(u)\Phi_{\Gamma,\widetilde{Y}}(u). \end{aligned} \tag{2.6}$$

(iii) (*Decomposition algorithm*)

$$\begin{aligned} (A(u),\; B(u)) &= R_0^{-1}(C)(u), \;\; \textit{i.e.,} \\ A(u) &= C(u)\Phi_{X',Y}(u)^*, \\ B(u) &= C(u)\Phi_{\Gamma',Y}(u)^*. \end{aligned} \tag{2.7}$$

Proof. (i)$\Longrightarrow$(ii): Suppose that (i) holds. Write (i) as $y = x + z$ such that $y \in V_1$, $x \in V_0$ and $z \in W_0$. Then by [10, Proposition 3.3],

$$\begin{aligned} C(u) &= \Phi_{\{y\},\widetilde{Y}}(u) \;\; = \;\; \Phi_{\{y\},\widetilde{S}}(u)\Phi_{\widetilde{Y},S}(u)^* \\ &= \left(\Phi_{\{y\},X'}(u) \;\; \Phi_{\{y\},\Gamma'}(u)\right) \Phi_{S,\widetilde{Y}}(u) \\ &= \left(\Phi_{\{x\},X'}(u) \;\; \Phi_{\{z\},\Gamma'}(u)\right) \Phi_{S,\widetilde{Y}}(u) \\ &= (A(u) \;\; B(u)) \; \Phi_{S,\widetilde{Y}}(u), \end{aligned}$$

since $z \perp U^{\mathbf{Z}^d}(X')$ and $x \perp U^{\mathbf{Z}^d}(\Gamma')$. Hence (ii) holds.

The proof of (ii)$\Longleftrightarrow$(iii) is obvious. Suppose that (ii) holds. Let

$$x = \sum_{j=1}^{r}\sum_{n\in\mathbf{Z}^d} a_j(n)U^n x_j \quad \text{and} \quad z = \sum_{k=1}^{s-r}\sum_{n\in\mathbf{Z}^d} b_k(n)U^n z_k.$$

Let $y = x + z$ be expressed as $\sum_{\ell=1}^{s}\sum_{n\in\mathbf{Z}^d} c_\ell'(n)U^n y_\ell$, where $c_\ell' \in \ell^2(\mathbf{Z}^d)$, $\ell = 1,\dots,s$. If $C_\ell'(u) = \sum_{n\in\mathbf{Z}^d} c_\ell'(n)e^{in\cdot u}$, $\ell = 1,\dots,s$, then by the implication (i)$\Longrightarrow$(ii) established above and (ii), $(C'_1(u)\ \dots\ C'_s(u)) = (A(u)\ \ B(u))\ \Phi_{S,\widetilde{Y}}(u) = C(u)$. Hence $c'_\ell = c_\ell,\ \ell = 1,\dots,s$, and so (i) holds. □

Corollary 2.5. *Suppose that the conditions in Theorem 2.1 hold. If*

$$y = \sum_{\ell=1}^{s} \sum_{n\in\mathbf{Z}^d} c_\ell(n) U^n y_\ell$$

and

$$C(u) = \left(\sum_{n\in\mathbf{Z}^d} c_1(n) e^{in\cdot u}\ \dots\ \sum_{n\in\mathbf{Z}^d} c_s(n) e^{in\cdot u} \right),$$

then

$$P_{V_0//W_0}(y) = \sum_{j=1}^{r} \sum_{n\in\mathbf{Z}^d} a_j(n) U^n x_j \quad \textit{and} \quad P_{W_0//V_0}(y) = \sum_{k=1}^{s-r} \sum_{n\in\mathbf{Z}^d} b_k(n) U^n z_k,$$

where

$$\left(\sum_{n\in\mathbf{Z}^d} a_1(n) e^{in\cdot u}\ \dots\ \sum_{n\in\mathbf{Z}^d} a_r(n) e^{in\cdot u} \right) = C(u)\, \Phi_{X',Y}(u)^*$$

and

$$\left(\sum_{n\in\mathbf{Z}^d} b_1(n) e^{in\cdot u}\ \dots\ \sum_{n\in\mathbf{Z}^d} b_{s-r}(n) e^{in\cdot u} \right) = C(u)\, \Phi_{\Gamma',Y}(u)^*.$$

3. Some connections between $L^2(\mathbf{T}^d)$ and dilation matrices

For the rest of this paper, let M be a $d\times d$ matrix with integer entries such that $m = |det(M)| \geq 2$. Such a matrix is often called a *dilation* matrix. Let M^T be the transpose of M. Let $\mathcal{C}_M = \{\gamma_0, \gamma_1, \dots, \gamma_{m-1}\}$ (respectively $\mathcal{C}_{M^T} = \{\eta_0, \eta_1, \dots, \eta_{m-1}\}$) be a full set of coset representatives of $\mathbf{Z}^d/M\mathbf{Z}^d$ (respectively $\mathbf{Z}^d/M^T\mathbf{Z}^d$), where $\gamma_0 = \eta_0 = 0$. Then $\mathbf{Z}^d$ is the disjoint union of the cosets $M\mathbf{Z}^d + \gamma_j, j = 0, 1, \dots, m-1$ (respectively $M^T\mathbf{Z}^d + \eta_j, j = 0, 1, \dots, m-1$).

It is well known that the following orthogonality relations hold:

$$\sum_{k=0}^{m-1} e^{i2\pi(\eta_j-\eta_\ell)\cdot M^{-1}\gamma_k} = m\,\delta_{j,\ell}\,, \quad j,\ell = 0,1,\dots,m-1, \tag{3.1}$$

and

$$\sum_{j=0}^{m-1} e^{i2\pi\eta_j\cdot M^{-1}(\gamma_k-\gamma_\ell)} = m\,\delta_{k,\ell}\,, \quad k,\ell = 0,1,\dots,m-1. \tag{3.2}$$

Let

$$Q(u) = \frac{1}{\sqrt{m}} \left(e^{i\gamma_k\cdot u}\, e^{i2\pi\eta_\ell\cdot M^{-1}\gamma_k} I_r \right)_{k,\ell=0}^{m-1}, \quad u \in \mathbf{R}^d, \tag{3.3}$$

where I_r is the $r\times r$ identity matrix. By (3.1) and (3.2), $Q(u)$ is a unitary $mr\times mr$ matrix (indeed the product of two unitary matrices).

Let q be a fixed positive integer and let the map

$$J : \underbrace{L^2(\mathbf{T}^d)^{q\times r} \oplus \ldots \oplus L^2(\mathbf{T}^d)^{q\times r}}_{m\ times} \longrightarrow L^2(\mathbf{T}^d)^{q\times r}$$

be defined by

$$\begin{aligned}J(G_0, \ldots , G_{m-1})(u) &= \left(G_0(M^T u)\ \ldots\ G_{m-1}(M^T u)\right)\, \Omega(u) \\ &= G_0(M^T u) + G_1(M^T u)e^{i\gamma_1\cdot u} + \ldots + G_{m-1}(M^T u)e^{i\gamma_{m-1}\cdot u} \\ &= \sum_{j=0}^{m-1} \sum_{n\in\mathbf{Z}^d} g_j(n)e^{i(Mn+\gamma_j)\cdot u},\end{aligned} \tag{3.4}$$

where $G_j(u) = \sum_{n\in\mathbf{Z}^d} g_j(n)e^{in\cdot u}$, $g_j \in \ell^2(\mathbf{Z}^d)^{q\times r}$, $j = 0, 1, ..., m-1$, and

$$\Omega(u) = \begin{pmatrix} I_r \\ e^{i\gamma_1\cdot u} I_r \\ \vdots \\ e^{i\gamma_{m-1}\cdot u} I_r \end{pmatrix}, \quad u \in \mathbf{R}^d. \tag{3.5}$$

Theorem 3.1. *The map J is a unitary operator, and its inverse*

$$J^* : L^2(\mathbf{T}^d)^{q\times r} \longrightarrow \underbrace{L^2(\mathbf{T}^d)^{q\times r} \oplus \ldots \oplus L^2(\mathbf{T}^d)^{q\times r}}_{m\ times}$$

is given by $J^*(G)(u) = (F_0(u),\ \ldots\ ,\ F_{m-1}(u))$, where

$$G(u) = \sum_{n\in\mathbf{Z}^d} g(n)e^{in\cdot u},\ \ F_j(u) = \sum_{n\in\mathbf{Z}^d} g(Mn + \gamma_j)e^{in\cdot u},\ \ j = 0, 1, ..., m-1. \tag{3.6}$$

Proof. Since

$$\|J(G_0, \ldots , G_{m-1})\|^2 = \sum_{j=0}^{m-1} \sum_{n\in\mathbf{Z}^d} \|g_j(n)\|^2 = \sum_{j=0}^{m-1} \|G_j\|^2,$$

the map J is isometric. For any G in $L^2(\mathbf{T}^d)^{q\times r}$,

$$\begin{aligned}G(u) &= \sum_{n\in\mathbf{Z}^d} g(n)e^{in\cdot u} = \sum_{j=0}^{m-1} \sum_{n\in\mathbf{Z}^d} g(Mn + \gamma_j)e^{i(Mn+\gamma_j)\cdot u} \\ &= \sum_{j=0}^{m-1} F_j(M^T u)e^{i\gamma_j\cdot u},\end{aligned}$$

where F_j, $j = 0, 1, ..., m-1$, are given by (3.6). By (3.4), $G = J(F_0, \ldots , F_{m-1})$. Therefore J is surjective, and the inverse of J has the desired form. □

For any G in $L^2(\mathbf{T}^d)^{q\times r}$, define a function G_{mod} in $L^2(\mathbf{T}^d)^{q\times mr}$ by

$$G_{mod}(u) = \frac{1}{\sqrt{m}} \left(G(u)\ \ G(u + 2\pi(M^T)^{-1}\eta_1)\ \ldots\ G(u + 2\pi(M^T)^{-1}\eta_{m-1})\right). \tag{3.7}$$

Remark 3.2. $Q(u) = \Omega_{mod}(u)$, where $\Omega(u)$ is given by (3.5).

We summarize below some properties of G_{mod} which will be needed in the next section. We omit their proofs.

Proposition 3.3. *For any G in $L^2(\mathbf{T}^d)^{q\times r}$,*

(i) $G_{mod}(u) = (J^*G)(M^T u)\, Q(u)$,

(ii) $J^*(G)(u) = G_{mod}((M^T)^{-1}u)\, Q((M^T)^{-1}u)^*$,

(iii) $\displaystyle \frac{1}{m}\sum_{j=0}^{m-1} G(u + 2\pi(M^T)^{-1}\eta_j) = \sum_{n\in\mathbf{Z}^d} g(Mn)e^{iMn\cdot u}$, *where*

$$G(u) = \sum_{n\in\mathbf{Z}^d} g(n)e^{in\cdot u}.$$

Consider next the special case $q = mr$.

Proposition 3.4. *Let $G \in L^2(\mathbf{T}^d)^{mr\times r}$. Suppose that the $mr \times mr$ matrices $G_{mod}(u)$ are invertible for almost all u in $\mathbf{R}^d$ and G_{mod}^{-1} is in $L^2(\mathbf{T}^d)^{mr\times mr}$. Let*

$$\tilde{G}(u) = \sqrt{m}\,(G_{mod}(u)^{-1})^* \begin{pmatrix} I_r \\ 0_r \\ \vdots \\ 0_r \end{pmatrix}_{mr\times r}, \quad u \in \mathbf{R}^d. \tag{3.8}$$

Then $\tilde{G}$ is in $L^2(\mathbf{T}^d)^{mr\times r}$ and $G_{mod}(u)^{-1} = \tilde{G}_{mod}(u)^*$.

4. Characterizations of oblique multiwavelets

We shall follow the same notations as in sections 1 and 3. As before, let $U = (U_1, ..., U_d)$ be an ordered d-tuple of distinct commuting unitary operators on a Hilbert space H. Let D be another unitary operator on H such that

$$U^n D = DU^{Mn}, \quad n \in \mathbf{Z}^d\ , \tag{4.1}$$

where M is a $d \times d$ matrix as in section 3. Then for every n in $\mathbf{Z}^d$, there exist a unique ℓ in $\{0, 1, \ldots, m-1\}$ and a unique p in $\mathbf{Z}^d$ such that $n = Mp + \gamma_\ell$. Hence

$$DU^n = U^p DU^{\gamma_\ell}. \tag{4.2}$$

Let $X = \{x_1, ..., x_r\}$ be a finite subset of H such that

$$\{\langle x_j, U^n x_k\rangle\}_{n\in\mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d), \quad j, k = 1, \ldots, r. \tag{4.3}$$

Let $L = \{(\ell, j) : \ \ell = 0, 1, \ldots, m-1,\ j = 1, \ldots, r\}$ with *lexicographical* ordering in (ℓ, j),

$$y_{\ell,j} = DU^{\gamma_\ell} x_j, \quad (\ell, j) \in L, \tag{4.4}$$

and let $Y = \{y_{\ell,j} : \ (\ell, j) \in L\}$. Let

$$V_0 = \langle U^{\mathbf{Z}^d}(X)\rangle \quad \text{and} \quad V_1 = \langle U^{\mathbf{Z}^d}(Y)\rangle. \tag{4.5}$$

For the time being, we do *not* assume that $V_0 \subset V_1$.

By (4.3), the function

$$\Phi_{X,X}(u) = \left(\sum_{n\in\mathbf{Z}^d} \langle x_j, U^n x_k\rangle e^{in\cdot u}\right)_{1\le j\le r,\, 1\le k\le r} \tag{4.6}$$

is in $L^2(\mathbf{T}^d)^{r\times r}$ and the function

$$\Phi_{Y,Y}(u) = \left(\sum_{n\in\mathbf{Z}^d} \langle y_{\ell,j}, U^n y_{p,k}\rangle e^{in\cdot u}\right)_{(\ell,j)\in L,\,(p,k)\in L} \tag{4.7}$$

is in $L^2(\mathbf{T}^d)^{mr\times mr}$. Define a function $\Phi^{[m]}_{X,X}$ in $L^2(\mathbf{T}^d)^{mr\times mr}$ by

(4.8)

$$\Phi^{[m]}_{X,X}(u) = \begin{pmatrix} \Phi_{X,X}(u) & 0_r & \dots & 0_r \\ 0_r & \Phi_{X,X}(u+2\pi(M^T)^{-1}\eta_1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_r \\ 0_r & \dots & 0_r & \Phi_{X,X}(u+2\pi(M^T)^{-1}\eta_{m-1}) \end{pmatrix}.$$

Theorem 4.1. *We have the following relations between X and Y.*

(i) $V_1 = D(V_0)$.

(ii) $\Phi_{Y,Y}(M^T u) = Q(u)\,\Phi^{[m]}_{X,X}(u) Q(u)^*$.

(iii) *$U^{\mathbf{Z}^d}(X)$ is a Riesz basis for V_0 if and only if $U^{\mathbf{Z}^d}(Y)$ is a Riesz basis for V_1.*

Proof. (i) and (iii) follow easily from (4.2) and the assumption that D is unitary.

By (4.1) and (4.4), for $t, p = 0, 1, \dots, m-1,\ j, k = 1, \dots, r$,

$$\langle y_{t,j}, U^n y_{p,k}\rangle = \langle DU^{\gamma_t} x_j, DU^{Mn+\gamma_p} x_k\rangle = \langle x_j, U^{Mn+\gamma_p-\gamma_t} x_k\rangle.$$

Hence the (t,p)-block of $\Phi_{Y,Y}(M^T u)$ is

$$\left(\sum_{n\in\mathbf{Z}^d} \langle y_{t,j}, U^n y_{p,k}\rangle e^{in\cdot M^T u}\right)^r_{j,k=1} = \sum_{n\in\mathbf{Z}^d} \left(\langle x_j, U^{Mn+\gamma_p-\gamma_t} x_k\rangle\right)^r_{j,k=1} e^{iMn\cdot u}.$$

Using (3.3), the (t,p)-block of $Q(u)\,\Phi^{[m]}_{X,X}(u)\,Q(u)^*$ is

$$\begin{aligned} (*) \quad &= \frac{1}{m}\sum_{\ell=0}^{m-1} e^{i(\gamma_t\cdot u+2\pi\eta_\ell\cdot M^{-1}\gamma_t)}\,\Phi_{X,X}(u+2\pi(M^T)^{-1}\eta_\ell)\,e^{-i(\gamma_p\cdot u+2\pi\eta_\ell\cdot M^{-1}\gamma_p)} \\ &= \frac{1}{m}\sum_{\ell=0}^{m-1} G(u+2\pi(M^T)^{-1}\eta_\ell), \end{aligned}$$

where

$$\begin{aligned} G(u) &= \Phi_{X,X}(u)e^{i(\gamma_t-\gamma_p)\cdot u} = \left(\sum_{n\in\mathbf{Z}^d} \langle x_j, U^n x_k\rangle e^{i(n+\gamma_t-\gamma_p)\cdot u}\right)^r_{j,k=1} \\ &= \sum_{n\in\mathbf{Z}^d} \left(\langle x_j, U^{n+\gamma_p-\gamma_t} x_k\rangle\right)^r_{j,k=1} e^{in\cdot u}. \end{aligned}$$

Hence by Proposition 3.3,

$$(*) = \sum_{n\in\mathbf{Z}^d} \left(\langle x_j, U^{Mn+\gamma_p-\gamma_t} x_k\rangle\right)^r_{j,k=1} e^{iMn\cdot u},$$

which is the (t,p)-block of $\Phi_{Y,Y}(M^T u)$. Hence (ii) holds. □

Henceforth, assume that $U^{\mathbf{Z}^d}(X)$ is a Riesz basis for V_0 so that $U^{\mathbf{Z}^d}(Y)$ is a Riesz basis for $V_1 = D(V_0)$. Let $\widetilde{X} = \{\tilde{x}_1, ..., \tilde{x}_r\}$ be a subset of V_0 such that $U^{\mathbf{Z}^d}(\widetilde{X})$ is a Riesz basis for V_0 biorthogonal to $U^{\mathbf{Z}^d}(X)$. Since D is unitary, $\{DU^n x_j : n \in \mathbf{Z}^d, j = 1, ..., r\}$ and $\{DU^n \tilde{x}_j : n \in \mathbf{Z}^d, j = 1, ..., r\}$ are biorthogonal Riesz bases for V_1. Let

$$\tilde{y}_{\ell,j} = DU^{\gamma_\ell}\tilde{x}_j, \quad (\ell, j) \in L, \tag{4.9}$$

and let $\widetilde{Y} = \{\tilde{y}_{\ell,j} : (\ell, j) \in L\}$. It is not difficult to show that $U^{\mathbf{Z}^d}(Y)$ and $U^{\mathbf{Z}^d}(\widetilde{Y})$ are biorthogonal Riesz bases for V_1.

Let $W = \{w_1, ..., w_q\}$ be a subset of V_1, for some positive integer q. Then

$$w_k = \sum_{j=1}^{r} \sum_{n\in\mathbf{Z}^d} g_{k,j}(n) DU^n x_j, \quad k = 1, ..., q, \tag{4.10}$$

where $g_{k,j} \in \ell^2(\mathbf{Z}^d),\ k = 1, ..., q,\ j = 1, ..., r$. Let

$$g(n) = (g_{k,j}(n))_{1\le k\le q,\, 1\le j\le r}\,, \quad n \in \mathbf{Z}^d. \tag{4.11}$$

Then g is in $\ell^2(\mathbf{Z}^d)^{q\times r}$ and (4.10) can be written in matrix form as

$$\begin{pmatrix} w_1 \\ \vdots \\ w_q \end{pmatrix} = \sum_{n\in\mathbf{Z}^d} g(n) \begin{pmatrix} DU^n x_1 \\ \vdots \\ DU^n x_r \end{pmatrix}. \tag{4.12}$$

Define a function G in $L^2(\mathbf{T}^d)^{q\times r}$ by

$$G(u) = \sum_{n\in\mathbf{Z}^d} g(n) e^{in\cdot u}, \quad u \in \mathbf{R}^d. \tag{4.13}$$

Theorem 4.2. *The function G associated with W has the following properties.*

(i) $\Phi_{W,\widetilde{Y}}(u) = J^*(G)(u)$.

(ii) $G_{mod}(u) = \Phi_{W,\widetilde{Y}}(M^T u)\, Q(u)$.

(iii) *$U^{\mathbf{Z}^d}(W)$ is a Riesz basis for $\langle U^{\mathbf{Z}^d}(W)\rangle$ if and only if there exist positive constants C_1 and C_2 such that*

$$C_1 \le G_{mod}(u)\, G_{mod}(u)^* \le C_2 \quad a.e.$$

Proof. (i): By (4.10), (4.2), (4.1) and (4.4), for $k = 1, ..., q$,

$$\begin{aligned} w_k &= \sum_{j=1}^{r} \sum_{n\in\mathbf{Z}^d} \sum_{\ell=0}^{m-1} g_{k,j}(Mn + \gamma_\ell) DU^{Mn+\gamma_\ell} x_j \\ &= \sum_{\ell=0}^{m-1} \sum_{j=1}^{r} \sum_{n\in\mathbf{Z}^d} g_{k,j}(Mn + \gamma_\ell) U^n y_{\ell,j}\,. \end{aligned}$$

Therefore for $\ell = 0, 1, ..., m-1$ and $j = 1, ..., r$,

$$\sum_{n\in\mathbf{Z}^d} \langle w_k, U^n \tilde{y}_{\ell,j}\rangle\, e^{in\cdot u} = \sum_{n\in\mathbf{Z}^d} g_{k,j}(Mn + \gamma_\ell) e^{in\cdot u}.$$

Hence by Theorem 3.1,

$$\Phi_{W,\widetilde{Y}}(u) = \left(\sum_{n\in\mathbf{Z}^d} g(Mn)e^{in\cdot u} \ \sum_{n\in\mathbf{Z}^d} g(Mn+\gamma_1)e^{in\cdot u} \ \dots \ \sum_{n\in\mathbf{Z}^d} g(Mn+\gamma_{m-1})e^{in\cdot u} \right)$$
$$= J^*(G)(u).$$

(ii) follows from (i) and Proposition 3.3.

By [10, Proposition 3.4], $U^{\mathbf{Z}^d}(W)$ is a Riesz basis for $\langle U^{\mathbf{Z}^d}(W)\rangle$ if and only if there exist positive constants C_1 and C_2 such that

$$C_1 \le \Phi_{W,\widetilde{Y}}(u)\, \Phi_{W,\widetilde{Y}}(u)^* \le C_2 \quad a.e.$$

Then (iii) follows from the relation

$$G_{mod}(u)\, G_{mod}(u)^* = \Phi_{W,\widetilde{Y}}(u)\, \Phi_{W,\widetilde{Y}}(u)^*\,, \tag{4.14}$$

which is obvious from (ii). □

We shall now see the effect of the change of bases for V_1.

Corollary 4.3. *Let a vector y in V_1 be expressed as*

$$y = \sum_{j=1}^{r} \sum_{n\in\mathbf{Z}^d} a_j(n) DU^n x_j = \sum_{\ell=1}^{m-1} \sum_{j=1}^{r} \sum_{n\in\mathbf{Z}^d} c_{\ell,j}(n) U^n y_{\ell,j},$$

where a_j and $c_{\ell,j}$ are in $\ell^2(\mathbf{Z}^d)$, $j = 1, ..., r$, $\ell = 0, 1, ..., m-1$. Let

$$A_j(u) = \sum_{n\in\mathbf{Z}^d} a_j(n)e^{in\cdot u}, \ \ j = 1, ..., r,$$
$$C_{\ell,j}(u) = \sum_{n\in\mathbf{Z}^d} c_{\ell,j}(n)e^{in\cdot u}, \ \ \ell = 0, 1, ..., m-1, \ j = 1, ..., r,$$

and

$$A = (A_1 \ \dots \ A_r), \ \ C = ((C_{0,j})_{j=1,\dots,r} \ \dots \ (C_{m-1,j})_{j=1,\dots,r})\,.$$

Then $C = J^*(A)$.

Proof. By Theorem 4.2(i), $C(u) = \Phi_{\{y\},\widetilde{Y}}(u) = J^*(A)(u)$. □

Corollary 4.4. *Let $W = \{w_1, ..., w_q\}$ and $\Gamma = \{z_1, ..., z_p\}$ be subsets of V_1 such that $\{\langle w_k, U^n z_j\rangle\}_{n\in\mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d)$, $\quad k = 1, \dots, q$, $j = 1, \dots, p$, and*

$$\begin{pmatrix} w_1 \\ \vdots \\ w_q \end{pmatrix} = \sum_{n\in\mathbf{Z}^d} g(n) \begin{pmatrix} DU^n x_1 \\ \vdots \\ DU^n x_r \end{pmatrix}, \quad \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix} = \sum_{n\in\mathbf{Z}^d} f(n) \begin{pmatrix} DU^n x_1 \\ \vdots \\ DU^n x_r \end{pmatrix},$$

where g is in $\ell^2(\mathbf{Z}^d)^{q\times r}$ and f is in $\ell^2(\mathbf{Z}^d)^{p\times r}$. Define

$$G(u) = \sum_{n\in\mathbf{Z}^d} g(n)e^{in\cdot u}, \ \ F(u) = \sum_{n\in\mathbf{Z}^d} f(n)e^{in\cdot u}, \ \ u \in \mathbf{R}^d.$$

Then

$$\Phi_{W,\Gamma}(M^T u) = G_{mod}(u)\, \Phi^{[m]}_{X,X}(u)\, F_{mod}(u)^* \tag{4.15}$$

$$= \frac{1}{m} \sum_{j=0}^{m-1} G(u + 2\pi(M^T)^{-1}\eta_j)\, \Phi_{X,X}(u + 2\pi(M^T)^{-1}\eta_j)\, F(u + 2\pi(M^T)^{-1}\eta_j)^*.$$

If moreover $U^{\mathbf{Z}^d}(X)$ *is orthonormal, then*

(i) $\Phi_{W,\Gamma}(M^T u) = G_{mod}(u)\, F_{mod}(u)^*$;

(ii) *assuming* $\{\langle w_k, U^n w_j\rangle\}_{n\in\mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d)$, $k,j = 1,\dots,q$, *then* $U^{\mathbf{Z}^d}(W)$ *is orthonormal if and only if* $G_{mod}(u)\, G_{mod}(u)^* = I_q$ *a.e.*

Proof. By [10, Proposition 3.3], Theorem 4.1 and Theorem 4.2,

$$\begin{aligned}\Phi_{W,\Gamma}(M^T u) &= \Phi_{W,\widetilde{Y}}(M^T u)\, \Phi_{Y,Y}(M^T u)\, \Phi_{\Gamma,\widetilde{Y}}(M^T u)^* \\ &= \Phi_{W,\widetilde{Y}}(M^T u) Q(u)\, \Phi^{[m]}_{X,X}(u)\, Q(u)^*\, \Phi_{\Gamma,\widetilde{Y}}(M^T u)^* \\ &= G_{mod}(u)\, \Phi^{[m]}_{X,X}(u)\, F_{mod}(u)^*.\end{aligned}$$

The rest of the assertions are straightforward. □

We now apply our previous results to the special case when $V_0 \subset V_1$. The next theorem, which is a consequence of Theorem 2.1 and Theorem 4.2, gives characterizations of oblique multiwavelets in the general setting of a Hilbert space.

Theorem 4.5. *Assume that*

$$V_0 \subset V_1. \tag{4.16}$$

Let $p = mr - r$, $\Gamma = \{z_1, \dots, z_p\} \subset V_1 \setminus V_0$,

$$\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = \sum_{n\in\mathbf{Z}^d} h(n) \begin{pmatrix} DU^n x_1 \\ \vdots \\ DU^n x_r \end{pmatrix}, \quad \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix} = \sum_{n\in\mathbf{Z}^d} g(n) \begin{pmatrix} DU^n x_1 \\ \vdots \\ DU^n x_r \end{pmatrix},$$

where h *is in* $\ell^2(\mathbf{Z}^d)^{r\times r}$ *and* g *is in* $\ell^2(\mathbf{Z}^d)^{p\times r}$. *Define*

$$H(u) = \sum_{n\in\mathbf{Z}^d} h(n) e^{in\cdot u}, \quad G(u) = \sum_{n\in\mathbf{Z}^d} g(n) e^{in\cdot u}, \quad u \in \mathbf{R}^d.$$

Let $S = X \cup \Gamma$ *and*

$$F(u) = \begin{pmatrix} H(u) \\ G(u) \end{pmatrix}, \quad u \in \mathbf{R}^d.$$

The following conditions are equivalent:

(i) $U^{\mathbf{Z}^d}(S)$ *is a Riesz basis for* V_1.

(ii) $U^{\mathbf{Z}^d}(\Gamma)$ *is a Riesz basis for* $W_0 := \langle U^{\mathbf{Z}^d}(\Gamma)\rangle$ *and* $V_0 \oplus W_0 = V_1$.

(iii) *There exist positive constants* C_1 *and* C_2 *such that*

$$C_1 \le F_{mod}(u)\, F_{mod}(u)^* \le C_2 \quad \text{a.e.}$$

(iv) *The matrices* $F_{mod}(u)$ *are invertible for almost all* u, *and the functions* $u \longrightarrow \|F_{mod}(u)\|$ *and* $u \longrightarrow \|F_{mod}(u)^{-1}\|$ *are essentially bounded.*

(v) *The operator* $R : L^2(\mathbf{T}^d)^r \oplus L^2(\mathbf{T}^d)^p \longrightarrow L^2(\mathbf{T}^d)^r$ *defined by*

$$\begin{aligned}R(A,B)(u) &= (A(M^T u)\;\; B(M^T u)) \begin{pmatrix} H(u) \\ G(u) \end{pmatrix} \\ &= A(M^T u) H(u) + B(M^T u) G(u)\end{aligned} \tag{4.17}$$

is bounded and invertible.

Proof. By Theorem 4.2,

$$\Phi_{S,\widetilde{Y}}(u) = J^*(F)(u), \tag{4.18}$$

$$F_{mod}(u) = \Phi_{S,\widetilde{Y}}(M^T u)\, Q(u), \tag{4.19}$$

and

$$F_{mod}(u)\, F_{mod}(u)^* \;= \Phi_{S,\widetilde{Y}}(M^T u)\, \Phi_{S,\widetilde{Y}}(M^T u)^* \,. \tag{4.20}$$

Next, by composing the operator $R_0 : L^2(\mathbf{T}^d)^r \oplus L^2(\mathbf{T}^d)^p \longrightarrow L^2(\mathbf{T}^d)^{mr}$ defined in (2.4) with the unitary operator $J : L^2(\mathbf{T}^d)^{mr} \longrightarrow L^2(\mathbf{T}^d)^r$ defined in (3.4) with $q = 1$, and using (4.18),

$$\begin{aligned} J\, R_0\,(A, B)(u) \;&=\; (A(M^T u)\;\; B(M^T u)) \begin{pmatrix} \Phi_{X,\widetilde{Y}}(M^T u) \\ \Phi_{\Gamma,\widetilde{Y}}(M^T u) \end{pmatrix} \Omega(u) \\ &=\; (A(M^T u)\;\; B(M^T u))\, F(u) \;=\; R(A, B)(u). \end{aligned}$$

Hence

$$R \,=\, J\, R_0. \tag{4.21}$$

By (4.20), (4.19) and (4.21), the equivalence of (i) to (v) in this theorem then follows easily from that in Theorem 2.1. □

Remark 4.6. Suppose that $U^{\mathbf{Z}^d}(X)$ is orthonormal. Then $U^{\mathbf{Z}^d}(S)$ is orthonormal if and only if $F_{mod}(u)$ are unitary for almost all $u \in \mathbf{R}^d$.

The next two results are analogues of Proposition 2.2 and Theorem 2.4 respectively.

Proposition 4.7. *Suppose that the conditions in Theorem 4.5 hold and* $p = mr - r$. *Let* $\tilde{F} \in L^2(\mathbf{T}^d)^{mr\times r}$ *be given as in Proposition 3.4 such that*

$$\tilde{F}_{mod}(u)^* \;= F_{mod}(u)^{-1} \,. \tag{4.22}$$

(i) *Then* $R^{-1} : L^2(\mathbf{T}^d)^r \longrightarrow L^2(\mathbf{T}^d)^r \oplus L^2(\mathbf{T}^d)^p = L^2(\mathbf{T}^d)^{mr}$ *is given by*

$$\begin{aligned} R^{-1}(C)(M^T u) \;&=\; C_{mod}(u)\tilde{F}_{mod}(u)^* \\ &=\; \frac{1}{m}\sum_{j=0}^{m-1} C(u + 2\pi(M^T)^{-1}\eta_j)\tilde{F}(u + 2\pi(M^T)^{-1}\eta_j)^*. \end{aligned} \tag{4.23}$$

(ii) *Write*

$$\tilde{F}(u) = \begin{pmatrix} \tilde{H}(u) \\ \tilde{G}(u) \end{pmatrix}, \;\; \text{where}\;\; \tilde{H}(u) = \sum_{n\in\mathbf{Z}^d} \tilde{h}(n)e^{in\cdot u}, \;\; \tilde{G}(u) = \sum_{n\in\mathbf{Z}^d} \tilde{g}(n)e^{in\cdot u}, \;\; u \in \mathbf{R}^d,$$

$\tilde{h}$ *is in* $\ell^2(\mathbf{Z}^d)^{r\times r}$ *and* $\tilde{g}$ *is in* $\ell^2(\mathbf{Z}^d)^{p\times r}$. *Let* $X' = \{x_1', ..., x_r'\}$ *and* $\Gamma' = \{z_1', ..., z_p'\}$ *be subsets of* V_1 *defined by*

$$\begin{pmatrix} x_1' \\ \vdots \\ x_r' \end{pmatrix} = \sum_{n\in\mathbf{Z}^d} \tilde{h}(n) \begin{pmatrix} DU^n\tilde{x}_1 \\ \vdots \\ DU^n\tilde{x}_r \end{pmatrix}, \quad \begin{pmatrix} z_1' \\ \vdots \\ z_p' \end{pmatrix} = \sum_{n\in\mathbf{Z}^d} \tilde{g}(n) \begin{pmatrix} DU^n\tilde{x}_1 \\ \vdots \\ DU^n\tilde{x}_r \end{pmatrix},$$

and let $\widetilde{S} = X' \cup \Gamma'$. *Then* $U^{\mathbf{Z}^d}(\widetilde{S})$ *is a Riesz basis for* V_1 *biorthogonal to* $U^{\mathbf{Z}^d}(S)$, $U^{\mathbf{Z}^d}(X')$ *biorthogonal to* $U^{\mathbf{Z}^d}(X)$, $U^{\mathbf{Z}^d}(\Gamma')$ *biorthogonal to* $U^{\mathbf{Z}^d}(\Gamma)$, $U^{\mathbf{Z}^d}(X')$ $\perp U^{\mathbf{Z}^d}(\Gamma)$ *and* $U^{\mathbf{Z}^d}(\Gamma') \perp U^{\mathbf{Z}^d}(X)$.

Proof. (i): Since $R = JR_0$, by Proposition 3.3, (4.19) and (4.22),

$$\begin{aligned} R^{-1}(C)(u) &= R_0^{-1}(J^*C)(u) \\ &= C_{mod}((M^T)^{-1}u)\, Q((M^T)^{-1}u)^*\, \Phi_{S,\widetilde{Y}}(u)^{-1} \\ &= C_{mod}((M^T)^{-1}u)\, F_{mod}((M^T)^{-1}u)^{-1} \\ &= C_{mod}((M^T)^{-1}u)\, \tilde{F}_{mod}((M^T)^{-1}u)^*. \end{aligned}$$

(ii): By Theorem 4.2 and Theorem 4.5 (with X and Y there replaced by $\widetilde{X}$ and $\widetilde{Y}$, respectively), $U^{\mathbf{Z}^d}(\widetilde{S})$ is a Riesz basis for V_1 and

$$\tilde{F}_{mod}(u) = \Phi_{\widetilde{S},Y}(M^T u)\, Q(u). \tag{4.24}$$

By [10, Proposition 3.3], (4.19), (4.24) and (4.22),

$$\Phi_{S,\widetilde{S}}(u) = \Phi_{S,\widetilde{Y}}(u)\, \Phi_{\widetilde{S},Y}(u)^* = F_{mod}((M^T)^{-1}u)\, \tilde{F}_{mod}((M^T)^{-1}u)^* = I_{mr} \ a.e.$$

Hence $U^{\mathbf{Z}^d}(\widetilde{S})$ is biorthogonal to $U^{\mathbf{Z}^d}(S)$. The rest of the assertions are obvious. □

Theorem 4.8. *Suppose that the conditions in Theorem 4.5 hold. Let* $p = mr - r$,

$$\begin{aligned} A_j(u) &= \sum_{n\in\mathbf{Z}^d} a_j(n)e^{in\cdot u}, \ \ a_j \in \ell^2(\mathbf{Z}^d), \ \ j = 1, ..., r, \\ B_k(u) &= \sum_{n\in\mathbf{Z}^d} b_k(n)e^{in\cdot u}, \ \ b_k \in \ell^2(\mathbf{Z}^d), \ \ k = 1, ..., p, \\ P_j(u) &= \sum_{n\in\mathbf{Z}^d} p_j(n)e^{in\cdot u}, \ \ p_j \in \ell^2(\mathbf{Z}^d), \ \ j = 1, ..., r, \end{aligned}$$

and

$$A = (A_1 \ \dots \ A_r), \ \ B = (B_1 \ \dots \ B_p), \ \ P = (P_1 \ \dots \ P_r).$$

The following conditions are equivalent:

(i) $$\sum_{j=1}^{r}\sum_{n\in\mathbf{Z}^d} p_j(n)DU^n x_j = \sum_{j=1}^{r}\sum_{n\in\mathbf{Z}^d} a_j(n)U^n x_j + \sum_{k=1}^{p}\sum_{n\in\mathbf{Z}^d} b_k(n)U^n z_k.$$

(ii) (*Reconstruction algorithm*)

$$\begin{aligned} P(u) &= R(A,B)(u) \\ &= A(M^T u)H(u) + B(M^T u)G(u). \end{aligned} \tag{4.25}$$

(iii) (*Decomposition algorithm*)

$$(A(u),\ B(u)) \ = \ R^{-1}(P)(u) \tag{4.26}$$

$$= \frac{1}{m}\sum_{j=0}^{m-1} P((M^T)^{-1}(u + 2\pi\eta_j))\tilde{F}((M^T)^{-1}(u + 2\pi\eta_j))^*,$$

where $\tilde{F}$ *is given in* (4.22).

Proof. Write

$$\sum_{j=1}^{r}\sum_{n\in\mathbf{Z}^d} p_j(n)DU^n x_j = \sum_{\ell=1}^{m-1}\sum_{j=1}^{r}\sum_{n\in\mathbf{Z}^d} c_{\ell,j}(n)U^n y_{\ell,j},$$

where $c_{\ell,j}$ are in $\ell^2(\mathbf{Z}^d)$, $j = 1, ..., r$, $\ell = 0, 1, ..., m-1$. Let

$$C_{\ell,j}(u) = \sum_{n\in\mathbf{Z}^d} c_{\ell,j}(n)e^{in\cdot u}, \ \ \ell = 0, 1, ..., m-1, \ j = 1, ..., r,$$

and $C = \left((C_{0,j})_{j=1,...,r} \ \ldots \ (C_{m-1,j})_{j=1,...,r}\right)$. By Corollary 4.3, $C = J^*(P)$. Then by Theorem 2.4,

$$\begin{aligned} \text{(i)} \iff & C = R_0(A,B) \\ \iff & J^*(P) = R_0(A,B) \\ \iff & P = JR_0(A,B) = R(A,B) \\ \iff & (A,B) \ = \ R^{-1}(P). \end{aligned}$$

□

Note that we can also obtain formulae for the projections $P_{V_0//W_0}$ and $P_{W_0//V_0}$, as in Corollary 2.5. We omit the details here.

Remark 4.9. (i) The results in this paper can be applied to the special case when $H = L^2(\mathbf{R}^d)$, $(U_kf)(x) = f(x - e_k)$, $k = 1, ..., d$, and $(Df)(x) = m^{\frac{1}{2}}f(Mx)$, for x in $\mathbf{R}^d$ and f in $L^2(\mathbf{R}^d)$, where $e_k = (\delta_{k,j})_{j=1,\ldots,d}$, $k = 1, \ldots, d$, M is a $d \times d$ dilation matrix considered in section 3 and $m = |det(M)| \geq 2$. In this case,

$$(DU^nf)(x) = m^{\frac{1}{2}}f(Mx - n), \quad n \in \mathbf{Z}^d,$$

and if $V = \{f_1, ..., f_r\}$ and $W = \{g_1, ..., g_p\}$ are subsets of $L^2(\mathbf{R}^d)$, then

$$\Phi_{V,W}(u) = \left(\sum_{n\in\mathbf{Z}^d} \hat{f}_k(u + 2\pi n)\,\overline{\hat{g}_j(u + 2\pi n)}\right)_{1\leq k\leq r, 1\leq j\leq p},$$

where $\hat{f}$ is the Fourier transform of a function f in $L^2(\mathbf{R}^d)$.

(ii) If $d = 1$, and U and D are unitary operators on a Hilbert space H such that $UD = DU^2$, using the notations in Theorem 4.5, then

$$F_{mod}(u) = \frac{1}{\sqrt{2}}\begin{pmatrix} H(u) & H(u+\pi) \\ G(u) & G(u+\pi) \end{pmatrix}.$$

In particular, Theorem 4.5 generalizes the characterizations of oblique multiwavelets given in [1, Theorem 3.1] and [2, Proposition 3.1, Theorem 3.2, Corollary 3.4], where the setting $H = L^2(\mathbf{R})$, $(Uf)(x) = f(x - 1)$ and $(Df)(x) = \sqrt{2}f(2x)$ is considered.

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