

STATISTICAL EXTENSIONS OF SOME CLASSICAL TAUBERIAN THEOREMS

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(Communicated by Albert Baernstein II)

ABSTRACT. Hardy's well-known Tauberian theorem for Cesàro means says that if the sequence x satisfies $\lim Cx = L$ and $\Delta x_k = O(1/k)$, then $\lim x = L$. In this paper it is shown that the hypothesis $\lim Cx = L$ can be replaced by the weaker assumption of the statistical limit: $\text{st-lim } Cx = L$, i.e., for every $\epsilon > 0$, $\lim_n \frac{1}{n} |\{k \leq n : |(Cx)_k - L| \geq \epsilon\}| = 0$. Similarly, the "one-sided" Tauberian theorem of Landau and Schmidt's Tauberian theorem for the Abel method are extended by replacing $\lim Cx$ and $\lim Ax$ with $\text{st-lim } Cx$ and $\text{st-lim } Ax$, respectively. The Hardy-Littlewood Tauberian theorem for Borel summability is also extended by replacing $\lim_t (Bx)_t = L$, where t is a continuous parameter, with $\lim_n (Bx)_n = L$, and further replacing it by $(B^*)\text{-st-lim } B^*x = L$, where B^* is the Borel matrix method.

1. INTRODUCTION

In [3] Fast defined statistical convergence of a complex number sequence x (denoted $\text{st-lim } x = L$) by

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0$$

for every $\epsilon > 0$, where the vertical bars denote the cardinality of the set which they enclose. In [5] a Tauberian theorem was given for statistical convergence:

if $\text{st-lim } x = L$ and $\Delta x_k = x_k - x_{k-1} = O(1/k)$, then $\lim x = L$.

NOTE: Throughout this paper Δx_k will denote the "backward difference" $x_k - x_{k-1}$, with $x_{-1} := 0$. The forward difference notation, which is frequently used in summability theory, is inconvenient when working with the one-sided Tauberian conditions.

Let Cx denote the Cesàro mean of order 1, i.e., $(Cx)_n = (1/n) \sum_{k=1}^n x_k$. In [6] Hardy presented one of the well-known Tauberian theorems for C :

Theorem 1.1. *If $\lim Cx = L$ and $\Delta x_k = O(1/k)$, then $\lim x = L$.*

Since it is clear that $\lim x = L$ implies $\text{st-lim } x = L$, it follows that $\text{st-lim } Cx = L$ is a weaker assumption than the hypothesis of Hardy's theorem. In Theorem 2.1 below we extend Hardy's theorem by using that weaker hypothesis.

Received by the editors March 5, 1998 and, in revised form, September 17, 1998.

1991 *Mathematics Subject Classification.* Primary 40E05.

Key words and phrases. Statistical convergence, Tauberian theorems.

In [11] Landau proved a stronger Tauberian theorem for C by replacing $\Delta x_k = O(1/k)$ with the condition: $k\Delta x_k \geq -c$ for some $c > 0$. We show that this one-sided condition is also a Tauberian condition for statistical convergence, and use that result to extend Landau's theorem by again replacing $\lim Cx = L$ with $\text{st-lim } Cx = L$. Hardy and Littlewood's Tauberian theorem for Borel summability [8] states that if x is Borel summable to L and $\Delta x_k = O(1/\sqrt{k})$, then $\lim x = L$. This theorem is extended similarly by replacing the hypothesis " x is Borel summable to L " with " $(B^*)\text{st-lim } (B^*x) = L$ ", where B^* denotes the Borel matrix method [13] (page 53), which includes B . A similar argument yields the statistical extension of the classical Tauberian theorem for the Euler-Knopp means [10].

2. TAUBERIAN THEOREMS FOR CESÀRO AND ABEL METHODS

In order to extend Hardy's "big oh" theorem for Cesàro means, we first prove that the order estimate of the differences is preserved under the Cesàro transformation.

Lemma 2.1. *If the sequence x satisfies $\Delta x_k = O(1/k)$, then $(\Delta Cx)_n = O(1/n)$.*

Proof. For $n > 1$ we have

$$\begin{aligned}
 n(\Delta Cx)_n &= n \left\{ \frac{1}{n} \sum_{k=1}^n x_k - \frac{1}{n-1} \sum_{k=1}^{n-1} x_k \right\} \\
 &= \frac{1}{n-1} \left\{ (n-1) \sum_{k=1}^n x_k - n \sum_{k=1}^{n-1} x_k \right\} \\
 &= \frac{1}{n-1} \left\{ (n-1)x_n - \sum_{k=1}^{n-1} x_k \right\} \\
 &= \frac{1}{n-1} \sum_{k=1}^{n-1} (x_n - x_k) \\
 &= \frac{1}{n-1} \sum_{k=1}^{n-1} \sum_{j=k+1}^n \Delta x_j \\
 &= \frac{1}{n-1} \sum_{j=2}^n (j-1) \Delta x_j \\
 &= \frac{1}{n-1} \sum_{j=2}^n O(1) \\
 &= O(1).
 \end{aligned}$$

□

Theorem 2.1. *If the sequence x satisfies $\text{st-lim } Cx = L$ and $\Delta x_k = O(1/k)$, then $\lim x = L$.*

Proof. By Lemma 2.1, $(\Delta Cx)_n = O(1/n)$, which, by Theorem 3 of [5], is a Tauberian condition for statistical convergence. Therefore, $\text{st-lim } Cx = L$ implies that $\lim Cx = L$. Now Hardy's theorem [6] yields $\lim x = L$. □

Our next theorem is an extension of Landau's theorem [11] that uses a one-sided Tauberian condition. An argument similar to that just used for Theorem 2.1

will suffice, but it is necessary to prove first a one-sided Tauberian condition for statistical convergence.

Theorem 2.2. *If the sequence x satisfies $st\text{-}\lim x = L$ and $k\Delta x_{k+1} \geq -c$, for some $c > 0$ and for every k , then $\lim x = L$.*

Proof. Assume $st\text{-}\lim x = L$ and use Theorem 1 of [5] to get a sequence y such that $\lim y = L$ and

$$(1) \quad \lim_n \frac{1}{n} |\{k \leq n : y_k \neq x_k\}| = 0.$$

Let $\{m(i)\}$ be the sequence of indices k for which $x_k \neq y_k$, and assume that $m(i) \rightarrow \infty$. (Otherwise it is trivial that $\lim x_k = L$.)

We now assert that $\limsup x \leq L$. If not, there is a positive number ϵ such that $x_{m(i)} \geq y_{m(i)} + 2\epsilon$ for infinitely many i . For each such i define $p(i)$ so that $x_{m(i)+p(i)}$ is the next term after $x_{m(i)}$ such that $x_{m(i)+p(i)} < y_{m(i)+p(i)} + \epsilon$. Thus it follows that

$$(2) \quad x_{m(i)} - x_{m(i)+p(i)} > \frac{\epsilon}{2},$$

for sufficiently large i . Since

$$\frac{|\{k \leq m(i) + p(i) : x_k \neq y_k\}|}{m(i) + p(i)} \geq \frac{p(i)}{m(i) + p(i)},$$

the limit (1) implies that

$$(3) \quad \lim_i \frac{p(i)}{m(i)} = 0.$$

Using $k\Delta x_{k+1} \geq -c$, we write

$$\begin{aligned} x_{m(i)} - x_{m(i)+p(i)} &= \sum_{k=m(i)+1}^{m(i)+p(i)} -\Delta x_k \\ &\leq \sum_{k=m(i)+1}^{m(i)+p(i)} \frac{c}{k-1} \\ &\leq \frac{cp(i)}{m(i)}. \end{aligned}$$

Since (3) implies that the last quantity tends to zero as $i \rightarrow \infty$, for sufficiently large i this contradicts (2), and we conclude that $\limsup x \leq L$.

To show that $\liminf x \geq L$, suppose to the contrary that $x_k \leq y_k - \epsilon$ for infinitely many k . We also have $x_i = y_i$ for infinitely many i , and each such i is followed by (a first) $m(i)$ where $x_{m(i)} < y_{m(i)} - \epsilon$. Therefore

$$(4) \quad x_i - x_{m(i)} = y_i - x_{m(i)} > y_i - (y_{m(i)} - \epsilon) > \frac{\epsilon}{2}$$

for i sufficiently large. As in the above case, the fact that $-\Delta x_k \leq c/(k-1)$ implies that $x_i - x_{m(i)}$ tends to zero, which contradicts (4). Hence, we must have $x_k > y_k - \epsilon$ for k sufficiently large, which implies that $\liminf x \geq \liminf y$, and the proof is complete. □

With the theorem just proved we can now give an extension of Landau's Tauberian theorem [11] for C .

Theorem 2.3. *If x is a sequence such that $st\text{-}\lim Cx = L$ and*

$$(5) \quad k\Delta x_{k+1} \geq -c, \text{ for some } c > 0 \text{ and for every } k,$$

then $\lim x = L$.

Proof. We first note that if x satisfies (5), then Cx also satisfies (5); for, by the proof of Lemma 2.1,

$$\begin{aligned} n(\Delta Cx)_n &= \frac{1}{n+1} \sum_{j=2}^{n+1} (j-1)\Delta x_j \\ &\geq \frac{1}{n+1} \sum_{j=2}^{n+1} (-c) \\ &= \frac{-nc}{n+1} \\ &\geq -c. \end{aligned}$$

Thus the hypotheses of Theorem 2.2 are satisfied by Cx , so by Theorem 2.2, $\lim Cx = L$. Now Landau’s theorem [11] yields the conclusion that $\lim x = L$. \square

We now consider the Abel matrix methods $A_\lambda = [a_{nk}]$ defined by

$$a_{nk} = \frac{1}{\lambda(n)} \left(1 - \frac{1}{\lambda(n)}\right)^k, \quad k = 0, 1, 2, \dots,$$

where $\lambda(n)$ is a strictly increasing sequence of real numbers so that $\lambda(0) \geq 1$. It is natural to suspect that under some conditions on the sequence $\{\lambda(n)\}$, the Abel matrix methods should preserve the $O(1/n)$ difference order condition. The following lemma shows a bit more. It will be convenient for us to extend λ to be defined over the interval $[1, \infty)$ by making it linear and continuous over $[j, j + 1]$ for $j = 1, 2, \dots$.

Lemma 2.2. *Let $\lambda(n)$ be a strictly increasing sequence of real numbers tending to infinity such that*

$$\lim_{\delta \searrow 0} \lim_n \frac{\lambda(n + \delta n)}{\lambda(n)} = 1.$$

If the sequence x is in the domain of the Abel matrix method A_λ and is of slow decreasing type, i.e.,

$$\lim_{\delta \searrow 0} \liminf_n \min_{n \leq m < n(1+\delta)} (x_m - x_n) \geq 0,$$

then the transformed sequence $z_n := (A_\lambda x)_n$ is also of slow decreasing type.

Proof. In [1] Armitage and Maddox showed that if we let $y_k(p) = p^k/k$, then

$$\begin{aligned} (1-p) \sum_{k=0}^{\infty} x_k p^k &= \sum_{k=1}^{\infty} (x_k - x_{k-1}) p^k \\ &= a_0 + \sum_{k=1}^{\infty} t_k (y_k(p) - y_{k+1}(p)), \quad 0 < p < 1, \end{aligned}$$

where

$$t_k = \sum_{j=1}^k j(x_j - x_{j-1}) = \sum_{j=0}^k (x_k - x_j).$$

Armitage and Maddox further showed that if x_k obeys the slowly decreasing Tauberian condition given above, then t_k is bounded below by $-Hk$ for some positive number H . Let $p_n := 1 - \frac{1}{\lambda(n)}$. If the Abel transform of x is denoted by $A(p) = (Ax)_p$, then, for any $m \in [n, n + \delta n)$ and for some positive constant H , we have

$$\begin{aligned} A(p_m) - A(p_n) &= \sum_{k=1}^{\infty} t_k \int_{p_n}^{p_m} t^{k-1}(1-t) dt \\ &\geq -H \sum_{k=1}^{\infty} k \int_{p_n}^{p_m} t^{k-1}(1-t) dt \\ &= -H \int_{p_n}^{p_m} (1-t)^{-1} dt \\ &\geq -H \log \left(\frac{\lambda(n + \delta n)}{\lambda(n)} \right). \end{aligned}$$

Hence, we see that

$$\begin{aligned} &\lim_{\delta \searrow 0} \liminf_n \min_{n \leq m < n(1+\delta)} ((A_\lambda x)_m - (A_\lambda x)_n) \\ &\geq \lim_{\delta \searrow 0} \liminf_n \left(-H \log \left(\frac{\lambda(n + \delta n)}{\lambda(n)} \right) \right) \\ &= -H \lim_{\delta \searrow 0} \limsup_n \log \left(\frac{\lambda(n + \delta n)}{\lambda(n)} \right) \\ &= 0. \end{aligned}$$

Consequently, we see that the sequence $z_k = (A_\lambda x)_k$ obeys the slow decrease condition. □

Remark 2.1. The above proof can be modified easily to show that the one-sided and the two-sided “big oh” difference Tauberian conditions are also preserved by the Abel matrix methods when $\lambda(n) = n^\alpha$ for some $\alpha > 0$.

Our next theorem extends the classical Tauberian theorems of Hardy and Littlewood [9], [12].

Theorem 2.4. *Let A^* be the Abel matrix method A_λ for which $\lambda(n) = n^\alpha$, for some $\alpha > 0$. If the sequence x satisfies $st\text{-}\lim A^*x = L$ and $k\Delta x_{k+1} \geq -c$ for some $c > 0$ and for every k , then $\lim x = L$.*

Proof. By Lemma 2.2 and the above remark, we see that $(A^*x)_n$ obeys the one-sided Tauberian condition. By Theorem 2.2, we must have $\lim(A^*x)_n = L$. Now Theorem 5 of [1] implies that x is Abel summable to L , which then gives that $\lim x = L$. □

Remark 2.2. We should remark that for the sake of simplicity we stated and proved the above result for the one-sided Tauberian condition. The following result, which

is analogous to Theorem 2.4, may be proved for the slow decrease condition by using the statistical Tauberian results provided in [4]. This then extends the classical Tauberian theorem of Schmidt [14].

Theorem 2.5. *Let $\lambda(n)$ be a strictly increasing sequence of real numbers going to infinity such that*

$$\lim_{\delta \searrow 0} \lim_n \frac{\lambda(n + \delta n)}{\lambda(n)} = 1.$$

If the sequence x satisfies $\text{st-lim } A_\lambda x = L$ and x satisfies the slow decrease condition, then $\lim x = L$.

3. TAUBERIAN THEOREMS FOR THE BOREL METHOD

For our final theorem we present a statistical extension of Hardy and Littlewood's Tauberian theorem [8] for Borel summability. Recall that the sequence x is Borel summable to L provided that

$$\lim_{t \rightarrow \infty} e^{-t} \sum_{k=0}^{\infty} \frac{x_k t^k}{k!} = L.$$

In order to use the notion of density that is inherent in statistical convergence, we replace the sequence-to-function Borel method with the Borel matrix method. Let B^* be the summability matrix whose nk -th term is given by

$$b_{nk} = \frac{e^{-n} n^k}{k!}.$$

We first show that B^* preserves the appropriate order in the differences.

Lemma 3.1. *If the sequence x satisfies $\Delta x_k = O(k^{-1/2})$, then*

$$(\Delta B^* x)_n = O(n^{-1/2}).$$

Proof. It is easy to see that

$$|x_k| = O(k^{1/2}) = O(k),$$

and therefore, $\sum |x_k| n^k / k!$ converges for each n , whence $(B^* x)_n$ is defined. Next note that in B^* , the $(n+1)$ -th row is equal to the Cauchy product of the n -th row with the "1-th" row: more precisely,

$$\begin{aligned} \sum_{j=0}^k b_{n,k-j} b_{1,j} &= \sum_{j=0}^k e^{-n} \frac{n^{k-j}}{(k-j)!} e^{-1} \frac{1^j}{j!} \\ &= \frac{e^{-(n+1)}}{k!} \sum_{j=0}^k \frac{n^{k-j} k!}{(k-j)! j!} \\ &= \frac{e^{-(n+1)} (n+1)^k}{k!} \\ &= b_{n+1,k}. \end{aligned}$$

Since each row sum of B^* equals one, we can write

$$\begin{aligned}
 |(B^*x)_{n+1} - (B^*x)_n| &= \left| \sum_{i=0}^{\infty} \sum_{j=0}^i b_{1,j} b_{n,i-j} x_i - \sum_{k=0}^{\infty} b_{n,k} x_k \right| \\
 &= \left| \sum_{j=0}^{\infty} b_{1,j} \sum_{i=j}^{\infty} b_{n,i-j} x_i - \sum_{k=0}^{\infty} b_{n,k} x_k \right| \\
 &= \left| \sum_{j=0}^{\infty} b_{1,j} \sum_{k=0}^{\infty} b_{n,k} x_{k+j} - \sum_{k=0}^{\infty} b_{n,k} x_k \right| \\
 &= \left| \sum_{j=0}^{\infty} b_{1,j} \sum_{k=0}^{\infty} b_{n,k} (x_{k+j} - x_k) \right| \\
 &= \left| \sum_{j=0}^{\infty} b_{1,j} \sum_{k=0}^{\infty} b_{n,k} \sum_{i=1}^j \Delta x_{k+i} \right| \\
 &\leq \sum_{j=0}^{\infty} b_{1,j} \sum_{k=0}^{\infty} b_{n,k} \sum_{i=1}^j \frac{K}{\sqrt{k+i}} \\
 &\leq K \sum_{j=0}^{\infty} j b_{1,j} \sum_{k=0}^{\infty} b_{n,k} \frac{1}{\sqrt{k+1}}.
 \end{aligned}$$

We now assert that the last (inner) sum is $O(n^{-1/2})$. Verifying this will complete the proof since the outer sum equals 1. By the Cauchy-Bunyakowsky-Schwarz Inequality, we see that

$$\begin{aligned}
 \sum_{k=0}^{\infty} b_{n,k} \frac{\sqrt{n}}{\sqrt{k+1}} &= \sum_{k=0}^{\infty} \sqrt{b_{n,k}} \frac{\sqrt{n}}{\sqrt{k+1}} \sqrt{b_{n,k}} \\
 &\leq \left(\sum_{k=0}^{\infty} b_{n,k} \frac{n}{k+1} \right)^{1/2} \left(\sum_{k=0}^{\infty} b_{n,k} \right)^{1/2} \\
 &= \left(\sum_{k=0}^{\infty} b_{n,k} \frac{n}{k+1} \right)^{1/2} \\
 &= \left(\sum_{k=0}^{\infty} b_{n,k+1} \right)^{1/2} \\
 &\leq 1.
 \end{aligned}$$

□

The next result concerns a variation of statistical convergence based on B^* . It is convenient to rewrite the definition of st-lim $x = L$ as

$$\lim_n \sum_{k \leq n: |x_k - L| \geq \epsilon} \frac{1}{n} = 0$$

for every $\epsilon > 0$. The “ $1/n$ ” in the sum is the n, k -th term of the Cesàro matrix C for $k \leq n$. We now replace $1/n$ with the n, k -th term of B^* and define (B^*) st-lim

$x = L$ provided that

$$\lim_n \sum_{k: |x_k - L| \geq \epsilon} b_{nk} = 0$$

for every $\epsilon > 0$. The following theorem is the B^* -analog of Theorem 3 of [5].

Theorem 3.1. *If the sequence x satisfies $(B^*)st\text{-lim } x = L$ and $\Delta x_k = O(k^{-1/2})$, then $\lim x = L$.*

Proof. The argument is similar to Hardy’s proof of Theorem 137 in [7]. We assume without loss of generality that $L = 0$ and suppose to the contrary that $\lim x \neq 0$. Then there exists an $\epsilon > 0$ such that $\limsup_k |x_k| > 3\epsilon$. Choose a subsequence $\{x_{n(q)}\}$ satisfying $|x_{n(q)}| > 2\epsilon$ for each q . Since $|\Delta x_k| \leq K/\sqrt{k}$ for some $K > 0$, we claim that for $\gamma = \epsilon/K$ we have

$$|x_k| > \epsilon, \quad \text{for each } q \text{ and } n(q) \leq k \leq n(q) + \gamma\sqrt{n(q)}.$$

Indeed, for an arbitrary q and $k \in [n(q), n(q) + \gamma\sqrt{n(q)}]$, we note that

$$\begin{aligned} |x_k| &= \left| x_{n(q)} + \sum_{j=n(q)}^{k-1} \Delta x_{j+1} \right| \\ &\geq |x_{n(q)}| - \sum_{j=n(q)}^{k-1} |\Delta x_{j+1}| \\ &> 2\epsilon - \sum_{j=n(q)}^{k-1} \frac{K}{\sqrt{j}} \\ &\geq 2\epsilon - \frac{K(k - n(q))}{\sqrt{n(q)}} \\ &\geq 2\epsilon - \frac{K\gamma\sqrt{n(q)}}{\sqrt{n(q)}} \\ &= \epsilon. \end{aligned}$$

Now $(B^*)st\text{-lim } x = 0$ gives that

$$\begin{aligned} 0 &= \limsup_q \sum_{k: |x_k| > \epsilon} b_{n(q),k} \\ &\geq \limsup_q \sum_{k \in [n(q), n(q) + \gamma\sqrt{n(q)}]} b_{n(q),k} \\ &> 0. \end{aligned}$$

The last inequality follows by parts (4) and (5) of Hardy’s Theorem 137 of [7]. This contradiction completes the proof. □

We are now prepared to prove a statistical extension of Hardy and Littlewood’s Tauberian theorem [8] for Borel summability. In addition to Lemma 3.1 and Theorem 3.1, we shall also need the following result due to Bingham [2].

Lemma 3.2. *If the sequence x satisfies $\lim B^*x = L$ and $\Delta x_k = O(1/\sqrt{k})$, then $\lim x = L$.*

The preceding lemma, which is obviously the matrix extension of Hardy and Littlewood's theorem, is not stated in this form by Bingham. This statement, however, follows immediately from his remark at the top of page 222 of [2] and Theorem 3b on page 224.

Theorem 3.2. *If the sequence x satisfies $(B^*)st\text{-}\lim B^*x = L$ and $\Delta x_k = O(1/\sqrt{k})$, then $\lim x = L$.*

Proof. By Lemma 3.1 the order hypothesis implies that $(\Delta B^*x)_n = O(n^{-1/2})$, which allows us to apply Theorem 3.1 to B^*x to infer that $\lim B^*x = L$. Now Lemma 3.2 can be invoked to conclude that $\lim x = L$. \square

The authors would like to thank the referee for several suggestions that have improved the exposition of this work.

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