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ON BURGESS'S THEOREM AND RELATED PROBLEMS

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ABSTRACT. Let G be a graph. We determine all graphs which are G-like. We also prove that if G_i $(i = 1, 2, \ldots , m)$ are graphs, then in order that each G_i-like $(i = 1, 2, \ldots , m)$ continuum M be n-indecomposable for some $n = n(M)$ it is necessary and sufficient that if K is a graph, then K is not G_i-like for some integer i with $1 \leq i \leq m$. This generalizes a well known theorem of Burgess.

1. INTRODUCTION

In this paper we study the structures of graph-like graphs and the structures of a finitely-many-graphs-like continua. Namely, if G is a graph, we determine all graphs which are G-like. We also prove that if G_i $(i = 1, 2, \ldots , m)$ are graphs, then in order that each G_i-like $(i = 1, 2, \ldots , m)$ continuum M is n-indecomposable for some $n = n(M)$ it is necessary and sufficient that if K is a graph, then K is not G_i-like for some integer i with $1 \leq i \leq m$. This generalizes a well known result of Burgess. The results will be used in a forthcoming paper by the same authors in determining the set of periods of a piecewise monotone map of a graph (see [LXY] for some background).

By a *continuum* we mean a non-empty connected compact metric space. A continuum M is *decomposable* (resp., *indecomposable*) if it is (resp., is not) the union of its two proper subcontinua. Let X, Y be continua and d be a metric on X. A continuous surjective map $f : X \longrightarrow Y$ is an $\epsilon - map$ if for each $y \in Y$, $diam(f^{-1}(y)) < \epsilon$. If for each $\epsilon > 0$ there is an ϵ-map from X onto Y, then we say X is Y-like.

A continuum M is said to be the *essential sum* of some collection of its subcontinua if the union of the collection is M and there is no element of the collection such that it is contained in the union of the rest of the elements from the collection. If $n \in \mathbb{N}$ and the continuum M is the essential sum of n continua and it not the essential sum of $n + 1$ continua, then M is said to be $n - indecomposable$. It is known that for any such continuum M, there is a unique collection consisting of n indecomposable continua having M as their essential sum ([B1]).

By a *graph* we mean a connected compact one-dimensional branch manifold. Let G be a graph. For $x \in G$, there is a closed connected neighbourhood V of x such

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that if V' is a closed connected neighbourhood of x contained in V, then V' is homeomorphic to V. $\#(\partial(V))$ is denoted by $Val_G(x)$ and is called the *valence* of x, where $\partial(V)$ is the boundary of V in G and $\#(A)$ is the number of elements of the finite subset $A \subset G$. If $Val_G(x) = 1$, x is called an *end point* of G; if $Val_G(x) > 2$, x is called a *branch point* of G. We use $e(G)$ and $b(G)$ to denote the set of end points of G and the set of branch points of G respectively. A finite set $v(G) \supset b(G) \cup e(G)$ is the set of *vertices* of G if for each simple closed curve S in G, $S \cap v(G) \subset b(G) \cup e(G)$ when $\#(S \cap (b(G) \cup e(G))) \geq 3$ and $\#(S \cap v(G)) = 3$ when $\#(S \cap (b(G) \cup e(G))) < 3$, i.e. we add some artificial points with valence 2 as vertices. In this way each *edge* (the closure of some connected component of $G \setminus v(G)$) is homeomorphic to $[0, 1]$ and if I, J are two edges of G, then either $I \cap J = \emptyset$ or $I \cap J$ is a set consisting of one point. A *free arc* of G is a subset of G homeomorphic to $[0, 1]$ which does not intersect with $v(G)$. Let $E(G) = \#(e(G))$, $B(G) = \#(b(G))$ and $V(G) = \#(v(G))$.

A *tree* is a graph containing no simple closed curve. A *star* is a tree with only one branch point or an arc.

2. Graphs which are G-like

In this section we will determine all graphs which are a given-graph-like. As corollaries we show quasi-homeomorphic graphs are homeomorphic, and if a locally connected continuum is a given-graph-like, then the continuum is a graph, and hence generalize some result of [MS]. We start with the following definition.

Definition 2.1. Let G, K be graphs. We say that $K \leq G$ if there are pairwise disjoint subgraphs of G such that K is homeomorphic to the graph obtained by shrinking the subgraphs to points.

An immediate observation is

Remark 2.2. Let G, K be graphs. If $K \leq G$, then $E(K) + B(K) \leq E(G) + B(G)$.

With the above definition we now show the main result of the section.

Theorem 2.3. *Let G be a graph. Then a graph K is G-like if and only if $K \leq G$.*

Proof. Let d be a metric on K. First we show that if $K \leq G$, then K is G-like. Assume that K is the graph obtained by shrinking subgraphs $G_1, \dots, G_n$ (of G) to points.

Let $q : G \longrightarrow K$ be the quotient map and $q(G_i) = \{x_i\}$, $1 \leq i \leq n$. Take a connected closed small neighbourhood U_i of x_i which is homeomorphic to n_i-star, where $n_i = Val(x_i)$. Furthermore, take a connected closed small neighbourhood V_i of G_i which has n_i-end points. Let $\epsilon > 0$. Then an ϵ-map g_ϵ from K onto G can be obtained by taking the union of $g_\epsilon|_{K\setminus U_i} = q^{-1}|_{G\setminus V_i}$ with an easily constructed ϵ-map from U_i onto V_i, $1 \leq i \leq n$.

Now we prove that if K is G-like, then $K \leq G$.

Let n be the number of edges of K. In each edge E_i of K choose a free arc A_i. Let $l = V(G) + 1$ and

$$\epsilon_1 = \min\{d(A_i, A_j) : 1 \leq i < j \leq n\}, \quad \epsilon_2 = \min\{\frac{diam(A_i)}{2l} : 1 \leq i \leq n\}.$$

Let $0 < \epsilon < \min\{\epsilon_1, \epsilon_2\}$ and $g_\epsilon : K \longrightarrow G$ be an ϵ-map. By dividing A_i into $2l$ subintervals with length $\frac{diam(A_i)}{2l}$ we get that there is a subinterval A_i' such that $g_\epsilon(A_i')$ is a free arc of G.

Let $C_1, \dots, C_p$ be the closures of connected components of $K \setminus (\bigcup_{i=1}^n A'_i)$. Then we have

1. C_i is a star;
2. $G = \bigcup_{i=1}^p g_\epsilon(C_i) \cup \bigcup_{i=1}^n g_\epsilon(A'_i)$;
3. $g_\epsilon(C_i) \cap g_\epsilon(C_j) = \emptyset$ for $i \neq j$;
4. if $C_i \cap A'_j \neq \emptyset$, then $g_\epsilon(C_i) \cap g_\epsilon(A'_j)$ is non-empty and a proper subinterval (may be degenerate) of $g_\epsilon(A'_j)$. Moreover, $E(g_\epsilon(C_i)) \geq E(C_i)$;
5. $g_\epsilon(A'_i) \cap g_\epsilon(A'_j) = \emptyset$ for $i \neq j$.

Hence a homeomorphic copy of K can be obtained by shrinking $g_\epsilon(C_i)$, $1 \leq i \leq p$, to points. That is, $K \leq G$. ☐

Continua M_1 and M_2 are said to be *quasi-homeomorphic* if M_1 is M_2-like and M_2 is M_1-like. It is well known that there are quasi-homeomorphic continua which are not homeomorphic (see for instance [K]). Contrary to this situation we have

Corollary 2.4. *Let G and K be graphs. Then G and K are homeomorphic if and only if G and K are quasi-homeomorphic.*

Proof. Assume that G and K are quasi-homeomorphic. By Theorem 2.3 and Remark 2.2 we get that $E(K) + B(K) = E(G) + B(G)$. It is easy to say that G and K should be homeomorphic by Theorem 2.3. ☐

In [MS] the authors show that if a locally connected continuum M is arc-like (circle-like), then M is an arc (a circle). Generalizing this result we have

Theorem 2.5. *Let M be a locally connected continuum and G be a graph. Then M is G-like if and only if M is a graph and $M \leq G$.*

To prove it we need the following simple lemma and the definition of the order of a point in a continuum (see [N, pp. 141–142]).

Lemma 2.6. *Let G, K be graphs. If K is G-like, then there is an ϵ-map $f_\epsilon : K \longrightarrow G$ such that $f_\epsilon(b(K)) \subset b(G)$.*

Proof. If there is $b \in b(K)$ such that $f_{\epsilon_i}(b) \notin b(G)$ for $\epsilon_i \longrightarrow 0$, then the image of some n-star (a small closed connected neighbourhood of b with $n = Val(b)$) under f_{ϵ_i} is an arc. That is, n-star ($n \geq 3$) is arc-like. This is impossible by Theorem 2.3. Hence the lemma follows. ☐

Proof of Theorem 2.5. We need to show that if M is G-like, then M is a graph.

As M is locally connected, M is path connected. Assume the contrary. That is, M is not a graph. Then there are $n = B(G) + 1$ different points $x_1, \dots, x_n$ of M such that $Ord(x_i, M) \geq 3$ ([N, p. 144]). Then there are disjoint graphs $G_i \subset M$, $1 \leq i \leq n$, such that each G_i has at least one branch point and $x_i \in G_i$. Applying Lemma 2.6 we get that G has at least n branch points, a contradiction. ☐

3. A GENERALIZATION OF BURGESS'S THEOREM

A well known result in continuum theory is that if a continuum is both arc-like and circle-like, then M is indecomposable or 2-indecomposable. In this section we will generalize this result by considering the structure of G_i-like ($i = 1, \dots, m$) continuum M. It turns out that in order that M should be n-indecomposable for some $n = n(M) \in \mathbb{N}$, G_i ($i = 1, \dots, m$) must have no common "shape". To do this we need the following lemma.

Lemma 3.1. *Let T be a tree and be an essential sum of the subtrees of $\{T_1, \dots, T_m\}$ for some $m \in \mathbb{N}$. Then there are at most $\sum_{t\in b(T)} Val(t)$ elements of $\{T_1, \dots, T_m\}$ which contain some points of $b(T)$.*

Proof. Assume that d is a metric on T. For each $b \in b(T)$ let $S(b)$ be the union of edges of T containing b and $e(S(b)) = \{e_b^1, \dots, e_b^{k_b}\}$. Furthermore, let $\mathcal{A} = \{T_1, \dots, T_m\}$. For each e_b^i with $b \in b(T)$ and $1 \le i \le k_b$ choose $T_b^i \in \mathcal{A}$ such that

$$d(e_b^i, T_b^i) = \min\{d(e_b^i, S) : S \in \mathcal{A} \text{ and } S \text{ contains } b\}.$$

We claim that if $S \in \mathcal{A}$ and S contains some point of $b(T)$, then $S \subset \bigcup_{i=1}^{k_b} \bigcup_{b\in b(T)} T_b^i$.

Assume the contrary. That is, $S \not\subset \bigcup_{i=1}^{k_b} \bigcup_{b\in b(T)} T_b^i$. Then there is $x \in S \cap (T \setminus b(T))$ with $x \notin \bigcup_{i=1}^{k_b} \bigcup_{b\in b(T)} T_b^i$. Let $E = [v_1, v_2]$ be the edge of T containing x, and without loss of generality we assume that $v_1 \in S \cap b(T)$ and $v_2 = e_{v_1}^{i_0}$. By the choice of $T_{v_1}^{i_0}$ we have that $x \in T_{v_1}^{i_0}$, a contradiction. This proves the claim.

As T is the essential sum of $\mathcal{A}$, we have that there are at most $\sum_{t\in b(T)} Val(t)$ elements of $\mathcal{A}$ which contain some points of $b(T)$. □

Corollary 3.2. *Let T be a tree and an essential sum of the subtrees of $\{T_1, \dots, T_m\}$ with $m = k + \sum_{t\in b(T)\cup e(T)} Val(t)$ for some $k \in \mathbb{N}$. Then there are at least k elements of $\{T_1, \dots, T_m\}$ which are free arcs of T. Furthermore, there are at least $[(k+1)/2]$ pairwise disjoint free arcs from $\{T_1, \dots, T_m\}$, where $[*]$ is the integer part of $*$.*

Proof. The first conclusion is an immediate consequence of Lemma 3.1. And the second one can be proved easily by induction on k. □

Note that we will use $\lim\{X, f_i\}$ to denote the inverse limit space of $f_i : X \longrightarrow X$, $i \in \mathbb{N}$.

Theorem 3.3. *Let T be a tree and G be a graph such that no free arc of G separates G. If M is a continuum which is both T-like and G-like, then M is n-indecomposable for some $n \le n_0 = 2l + \sum_{t\in b(T)\cup e(T)} Val(t)$, where $l = B(G)$.*

Proof. Assume that M is an essential sum of subcontinua $M_1, \dots, M_{n_0+1}$. Let $M = \lim\{T, f_i\} = \lim\{G, g_i\}$, and $p_i : M \longrightarrow T$ and $q_i : M \longrightarrow G$ be the i-th projection, $i \in \mathbb{N}$. It is easy to see that for i large enough, T is an essential sum of subtrees of $\{p_i(M_1), \dots, p_i(M_{n_0+1})\}$. By Corollary 3.2 there are at least $l+1$ elements $\{p_i(M_{i_1}), \dots, p_i(M_{i_{l+1}})\}$ of $\{p_i(M_1), \dots, p_i(M_{n_0+1})\}$ which are pairwise disjoint free arcs. Hence $M_{i_1}, \dots, M_{i_{l+1}}$ are pairwise disjoint and each M_i separates M.

Thus for j large enough $\{q_j(M_{i_1}), \dots, q_j(M_{i_{l+1}})\}$ are pairwise disjoint. By the choice of l, there is $1 \le h \le l+1$ such that $q_j(M_{i_h})$ is a free arc of G for infinitely many j. By the assumption on G, $q_j(M_{i_h})$ does not separate G.

Let $N_1,\ N_2$ be the two connected components of $\bigcup_{i=1, i\ne h}^{n_0+1} M_i$ and $\epsilon < d(N_1, N_2)$. Choose j_0 such that q_j is an ϵ-map for $j \ge j_0$. As $q_j(M_{i_h})$ does not separate G, there exist $x \in N_1$ and $y \in N_2$ such that $q_j(x) = q_j(y)$, a contradiction. □

Corollary 3.4 (Burgess). *If a continuum is both arc-like and circle-like, then M is either indecomposable or the union of two indecomposable subcontinua.*

Proof. As $E([0,1]) = 2$ and $B(S^1) = 0$, the corollary follows from Theorem 3.3 immediately. □

The following remark and example demonstrate that our result is more general than the result of Burgess. Let X and Y be two topological spaces. A continuous map $f : X \longrightarrow Y$ is *null homotopic* provided that f is homotopic to a constant map from X into Y.

Remark 3.5. Let G be a graph such that no free arc of G separates G. If $f : G \longrightarrow G$ is a surjective map and f is null homotopic, then the inverse limit $M = \lim\{G, f\}$ is n-indecomposable for some $n \leq n_0$, where n_0 is the number defined in Theorem 3.3.

Proof. Let $\widetilde{G}$ be the universal cover of G. Then $\widetilde{G}$ is an infinite tree such that each connected compact subset of $\widetilde{G}$ is a finite tree. Let $p : \widetilde{G} \longrightarrow G$ be the covering projection. Since f is null homotopic, there is a lifting $L : G \longrightarrow \widetilde{G}$ with $p \circ L = f$. Put $T = L(G)$. Then T is a finite tree and $p(T) = f(G) = G$. Set $F = L \circ p$. Then $p \circ F = f \circ p$, $F \circ L = L \circ f$ and $F(T) = T$.

Set $p' = p|_T : T \longrightarrow G$, $F' = F|_T : T \longrightarrow T$ and $L' = L : G \longrightarrow T$. Let $L'_\infty : M \longrightarrow \lim\{T, F'\}$, $f_\infty : M \longrightarrow M$ and $p'_\infty : \lim\{T, F'\} \longrightarrow M$ be the induced maps (see [N, p. 26]). Then $p'_\infty \circ L'_\infty = f_\infty$. Since f_∞ is a homeomorphism, L'_∞ is injective. It is clear that L'_∞ is surjective. Hence L'_∞ is a homemorphism. Then M is T-like and hence M is both G-like and T-like. By Theorem 3.3, M is n-indecomposable for some $n \leq n_0$. ☐

Example. For $m \in \mathbb{N}$ and each $1 \leq i \leq m$, let K_i be the copy of the Knaster's indecomposable continuum and p be the end point of K. Let M be the one point union of (K_i, p), $i = 1, \ldots, m$. Then K is m-indecomposable, and K is m-od-like and G-like, where G is the one point union of m circles.

With the above preparation now we prove the main result of this section. Note that for any finite graphs $G_1, \ldots, G_m$ ($m \in \mathbb{N}$), there are many continua which are G_i-like for each i, since we can use inverse systems whose terms are G_i (each G_i appears infinitely many times) and arbitrary surjective maps between them.

Theorem 3.6. *Let G_i $(1 \leq i \leq m)$ be graphs. Then in order that each G_i-like $(i = 1, \ldots, m)$ continuum M be n-indecomposable for some $n = n(M) \in \mathbb{N}$ it is necessary and sufficient that if K is a graph, then K is not G_i-like for some integer i with $1 \leq i \leq m$.*

Proof. (Necessity) It is obvious.

(Sufficiency) It is easy to see that $m \geq 2$. If each G_i contains a simple closed curve, then by Theorem 2.3 S^1 is G_i-like, $i = 1, \ldots, m$. If each G_i is separated by some free arc, then by Theorem 2.3 $[0, 1]$ is G_i-like, $i = 1, \ldots, m$. Hence if there is no graph K which is G_i-like ($i = 1, \ldots, m$), then there are i_0, j_0 such that G_{i_0} is a tree and G_{j_0} is a graph such that each free arc of G_{j_0} does not separate G_{j_0}. According to Theorem 3.3, M is n-indecomposable for some $n = n(M) \in \mathbb{N}$. ☐

The following related problems remain open:

Question 1. Let T be a tree and G be a graph such that no free arc of G separates G. Let

$$N(T, G) = \{n : M \text{ is both } T\text{-like and } G\text{-like, and is } n\text{-indecomposable}\}.$$

Is it true that $N(T, G) = \{1, \ldots, n_0\}$ for some $n_0 \leq 2B(G) + \sum_{t \in b(T) \cup e(T)} Val(t)$? If not, determine $N(T, G)$.

Question 2. Let T be a tree and G be a graph such that no free arc of G separates G. Let M be a continuum which is T-like. Is it true that if M is n-indecomposable for some $n \in N(T,G)$, then M is G-like?

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