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OPEN SUBGROUPS OF G AND ALMOST PERIODIC FUNCTIONALS ON $A(G)$

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Abstract. Let G be a locally compact group and let $C^*_\delta(G)$ denote the C^*-algebra generated by left translation operators on $L^2(G)$. Let $AP(\hat{G})$ and $WAP(\hat{G})$ be the spaces of almost periodic and weakly almost periodic functionals on the Fourier algebra $A(G)$, respectively. It is shown that if G contains an open abelian subgroup, then (1) $AP(\hat{G}) = C^*_\delta(G)$ if and only if $AP(\hat{G})_c$ is norm dense in $AP(\hat{G})$; (2) $WAP(\hat{G})$ is a C^*-algebra if $WAP(\hat{G})_c$ is norm dense in $WAP(\hat{G})$, where X_c denotes the set of elements in X with compact support. In particular, for any amenable locally compact group G which contains an open abelian subgroup, G has the *dual Bohr approximation property* and $WAP(\hat{G})$ is a C^*-algebra.

1. Introduction

Let G be a locally compact group with the left regular representation λ_G and let $VN(G)$ denote the von Neumann algebra generated by λ_G. Let $A(G)$ and $B(G)$ be the Fourier and Fourier-Stieltjes algebras of G, respectively. Then $VN(G)$ can be identified with the Banach dual of $A(G)$ and is a $B(G)$-module under the action $\langle u \cdot T, v\rangle = \langle T, uv\rangle$ for $u \in B(G)$, $v \in A(G)$, and $T \in VN(G)$.

Inside $VN(G)$, there are several well-known C^*-algebras, such as the reduced group C^*-algebra $C^*_\rho(G)$, the C^*-algebra $C^*_\delta(G)$ generated by $\{\lambda_G(x) : x \in G\}$ (see Lau [9]), and the C^*-algebra $UC(\hat{G})$ of uniformly continuous functionals on $A(G)$ which is the norm closure of $A(G)\cdot VN(G)$ introduced by Granirer [7]. $VN(G)$ also contains some important closed $B(G)$-submodules. Dunkl and Ramirez [4] defined the space $AP(\hat{G})$ (resp. $WAP(\hat{G})$) of almost periodic (resp. weakly almost periodic) functionals on $A(G)$ consisting of $T \in VN(G)$ such that the map $A(G) \to VN(G)$, $u \mapsto u \cdot T$, is compact (resp. weakly compact). Replacing $A(G)$ by $B(G)$, Chou [3] also defined the spaces $AP_B(\hat{G})$ and $WAP_B(\hat{G})$, called $B(G)$-almost periodic and $B(G)$-weakly almost periodic functionals on $A(G)$, respectively.

Obviously, $C^*_\delta(G) \subseteq AP_B(\hat{G}) \subseteq AP(\hat{G}) \subseteq WAP(\hat{G})$. If G is amenable, $WAP(\hat{G}) \subseteq UC(\hat{G})$ (Granirer [7]), $AP_B(\hat{G}) = AP(\hat{G})$ and $WAP_B(\hat{G}) = WAP(\hat{G})$ (Chou [3]).

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It is also known that $AP(\hat{G}) = C^*_\delta(G)$ and $WAP(\hat{G})$ is a C^*-algebra when G is either a locally compact abelian group or a discrete amenable group (Dunkl and Ramirez [4] and Granirer [7]). Chou [3] showed that the same conclusion is also true whenever G is amenable and contains an open normal abelian subgroup. However, there exists an infinite *compact* group G such that $AP(\hat{G}) \neq C^*_\delta(G)$ (see Chou [3] and Rindler [10]). Concerning the spaces $AP_B(\hat{G})$ and $WAP_B(\hat{G})$, Chou proved that, if H is an open normal subgroup of G such that G/H is amenable, then (a) $AP_B(\hat{G}) = C^*_\delta(G)$ if $AP_B(\hat{H}) = C^*_\delta(H)$; (b) $AP_B(\hat{G})$ (resp. $WAP_B(\hat{G})$) is a C^*-algebra if so is $AP_B(\hat{H})$ (resp. $WAP_B(\hat{H})$) (see [3, Theorem 4.4 and Proposition 4.6] and Remark 3.7 in the sequel). In general, the following questions are still open (cf. [3], [6], [8], [11], etc.): 1) for what kinds of groups G, does $AP(\hat{G}) = C^*_\delta(G)$ hold (such groups G are said to have the *dual Bohr approximation property* by Chou [3], since, for abelian groups G, $C^*_\delta(G)$ is precisely the norm closure of the trigonometrical polynomials on $\hat{G}$); 2) whether $AP(\hat{G})$ and $WAP(\hat{G})$ are always C^*-algebras; 3) whether $AP(\hat{G}) \subseteq UC(\hat{G})$ when G is non-amenable.

In this paper, we show that, for certain class of locally compact groups, questions 1)–3) are closely related. For a subspace X of $VN(G)$, instead of the inclusion $X \subseteq UC(\hat{G})$, we consider the following stronger but more essential condition, called *Condition (CD)*, on X: elements of X with compact support are norm dense in X. It can be seen that $AP(\hat{G})$ and $WAP(\hat{G})$ satisfy *(CD)* if G is amenable. Let H be an open subgroup of G. It is shown that $AP(\hat{G})$ satisfies *(CD)* if and only if [$AP(\hat{H})$ satisfies *(CD)* and $AP(\hat{G}) = \overline{\text{span}}\{\lambda_G(G)AP(\hat{H})\}$] (this is also true for $WAP(\hat{G})$, $AP_B(\hat{G})$, and $WAP_B(\hat{G})$). Consequently, if G is a locally compact group containing an open abelian subgroup, then

(1) G has the dual Bohr approximation property if and only if $AP(\hat{G})$ satisfies *(CD)*;

(2) $WAP(\hat{G})$ is a C^*-algebra if $WAP(\hat{G})$ satisfies *(CD)*.

In particular, if G is amenable and contains an open abelian subgroup, then G has the dual Bohr approximation property and $WAP(\hat{G})$ is a C^*-algebra. Several other related results on $AP_B(\hat{G})$ and $WAP_B(\hat{G})$ are also obtained.

The discussion on when $AP(\hat{G})$ is generated by the translates of elements in $AP(\hat{H})$ plays a key role in our investigation, which is motived by Chou [3].

2. Preliminaries and notations

Throughout this paper, G is a locally compact group and G_d denotes the group G with the discrete topology. For $U \subseteq G$, 1_U is the characteristic function of U.

Let $P(G)$ be the set of positive definite continuous functions on G. Then the Fourier-Stieltjes algebra $B(G)$ is the linear span of $P(G)$ and is identified with the Banach dual of $C^*(G)$ (the group C^*-algebra of G). The Fourier algebra $A(G)$ is the closed ideal in $B(G)$ generated by elements in $B(G)$ with compact support and is identified with the predual of $VN(G)$. See Eymard [5] for more details on $A(G)$ and $B(G)$.

For $T \in VN(G)$, $\operatorname{supp} T$ (the support of T) is the subset of G such that $x \in \operatorname{supp} T$ iff $u \cdot T = 0$ implies $u(x) = 0$ for all $u \in A(G)$. The C^*-subalgebra $UC(\hat{G})$ of $VN(G)$ is also a closed $B(G)$-submodule of $VN(G)$ and coincides with the norm closure of $\{T \in VN(G);\ \operatorname{supp} T \text{ is compact}\}$ in $VN(G)$ (see Granirer [7], [8]). The spaces $AP(\hat{G})$, $WAP(\hat{G})$, $AP_{_B}(\hat{G})$, and $WAP_{_B}(\hat{G})$ are self-adjoint closed $B(G)$-submodules of $VN(G)$.

3. Open subgroups of G and closed $B(G)$-submodules of $VN(G)$

In this section, H is an open subgroup of G. For any function h on H, h° is the trivial extension of h to G, i.e., $h^\circ(x) = 0$ for $x \in G \setminus H$. $VN_H(G)$ is the von Neumann algebra generated by $\{\lambda_G(x) : x \in H\}$. Then $VN_H(G) = \{T \in VN(G) : \operatorname{supp} T \subseteq H\}$.

The following lemma seems well known, but we could not find it in the literature. For the sake of completeness, we would like to present it and include its proof here.

Lemma 3.1. *For each $u \in B(H)$, $u^\circ \in B(G)$ and $\|u\| \le \|u^\circ\| \le 2\|u\|$. Furthermore, if $u \in B(H)$ is hermitian, then $u^\circ \in B(G)$ is also hermitian and $\|u^\circ\| = \|u\|$.*

Proof. Clearly, if $u \in P(H)$, then $u^\circ \in P(G)$ and $\|u^\circ\| = \|u\|$. If $u \in B(H)$ is hermitian, using the Jordan decomposition, we have $u^\circ \in B(G)$ is hermitian and $\|u^\circ\| = \|u\|$. For general $u \in B(H)$, $u = u_1 + iu_2$ with u_1 and u_2 hermitian. The conclusion follows. ☐

For $T \in VN(H)$, following the notation in [5], we use T° to denote the operator in $VN_H(G)$ defined by $\langle T^\circ, u\rangle = \langle T, u|_H\rangle$, $u \in A(G)$.

Remark 3.2. For each $y \in H$, $[\lambda_H(y)]^\circ = \lambda_G(y)$. Also, for $u \in B(H)$, $w \in B(G)$, and $T \in VN(H)$, we have $[u \cdot T]^\circ = u^\circ \cdot T^\circ$, $w \cdot T^\circ = [(w|_H) \cdot T]^\circ$, and $\operatorname{supp}(T^\circ) = \operatorname{supp} T$.

Let $T \in VN(H)$. Then $T \in C^*_\rho(H)$ iff $T^\circ \in C^*_\rho(G)$ (see Eymard [5, (3.21)]). It is also known that $T \in UC(\hat{H})$ iff $T^\circ \in UC(\hat{G})$ (see Granirer [7, Proposition 10]). Chou [3, Lemma 4.2] proved that $T \in AP(\hat{H})$ iff $T^\circ \in AP(\hat{G})$, and $T \in WAP(\hat{H})$ iff $T^\circ \in WAP(\hat{G})$. The same conclusion is also true for the following subspaces.

Lemma 3.3. *Let $T \in VN(H)$. Then*

(1) *$T \in C^*_\delta(H)$ if and only if $T^\circ \in C^*_\delta(G)$;*
(2) *$T \in AP_{_B}(\hat{H})$ if and only if $T^\circ \in AP_{_B}(\hat{G})$;*
(3) *$T \in WAP_{_B}(\hat{H})$ if and only if $T^\circ \in WAP_{_B}(\hat{G})$.*

Proof. (1) $[C^*_\delta(H)]^\circ \subseteq C^*_\delta(G)$ follows from Remark 3.2. Conversely, suppose $T^\circ \in C^*_\delta(G)$. Then $T^\circ = \lim \sum_{i=1}^n c_i\lambda_G(x_i)$ for some $c_i \in \mathbb{C}$, and $x_i \in G$. Since $\operatorname{supp} T^\circ \subseteq H$, $T^\circ = 1_H \cdot T^\circ = \lim \sum_{i=1}^n c_i[1_H \cdot \lambda_G(x_i)] = \lim \sum_{x_i \in H} c_i\lambda_G(x_i) = \lim\, [\sum_{x_i \in H} c_i\lambda_H(x_i)]^\circ$. Note that $S \mapsto S^\circ$ is a linear isometry. Therefore, $T = \lim \sum_{x_i \in H} c_i\lambda_H(x_i) \in C^*_\delta(H)$.

(2) Let $T \in AP_{_B}(\hat{H})$. Let (u_n) be a bounded sequence in $B(G)$. Then $(u_n|_H)$ is bounded in $B(H)$ and hence $(u_n|_H \cdot T)_n$ has a norm convergent subsequence $(u_{n_k}|_H \cdot T)_k$. Since $[u|_H \cdot T]^\circ = u \cdot T^\circ$, $(u_{n_k} \cdot T^\circ)_k$ is norm convergent. Thus, $T^\circ \in AP_{_B}(\hat{G})$. Conversely, let $T^\circ \in AP_{_B}(\hat{G})$. Let (v_n) be a bounded sequence in $B(H)$. By Lemma 3.1, (v_n°) is bounded in $B(G)$. So, $(v_n^\circ \cdot T^\circ)$ has a norm convergent subsequence. But $v^\circ \cdot T^\circ = [v \cdot T]^\circ$. Therefore, $(v_n \cdot T)$ has a norm convergent subsequence. Hence, $T \in AP_{_B}(\hat{H})$.

(3) Note that the adjoint of the map $T \mapsto T^\circ$ is from $VN(G)^*$ *onto* $VN(H)^*$. An argument similar to that of (2) follows. ☐

Corollary 3.4. *Let X be one of the spaces $C^*_\delta(G)$, $C^*_\rho(G)$, $UC(\hat{G})$, $AP(\hat{G})$, $WAP(\hat{G})$, $AP_{_B}(\hat{G})$, and $WAP_{_B}(\hat{G})$. Let X_H denote the corresponding subspace of $VN(H)$. Then $[X_H]^\circ = X \cap VN_H(G)$.*

Next, we investigate when X is spanned by the translates of elements in X°_H.

Proposition 3.5. *Let X be one of the spaces $C^*_\delta(G)$, $C^*_\rho(G)$ and $UC(\hat{G})$. Let X_H denote the corresponding subspace of $VN(H)$. Then $X = \overline{span}\{\lambda_G(G)[X_H]^\circ\}$.*

Proof. Note that $\lambda_G(G)X \subseteq X$ and $[X_H]^\circ \subseteq X$ (Corollary 3.4). So, we have that $\overline{\text{span}}\{\lambda_G(G)[X_H]^\circ \subseteq X$. We shall prove that $X \subseteq \overline{\text{span}}\{\lambda_G(G)[X_H]^\circ\}$ for $X = UC(\hat{G})$. The proofs for $X = C^*_\delta(G)$ and $X = C^*_\rho(G)$ follow from similar arguments.

Let $T \in UC(\hat{G})$ with compact support K. Then there exist $x_1, \cdots, x_n \in G$ such that $K \subseteq \bigcup_{i=1}^n x_iH$. We may assume that $\{x_iH\}_i$ are pairwise disjoint. Then $\sum_{i=1}^n 1_{x_iH} = 1$ on $\bigcup_{i=1}^n x_iH$. So, $T = \sum_{i=1}^n (1_{x_iH}) \cdot T$. We only need show that $1_{xH} \cdot T \in \lambda_G(x)[UC(\hat{H})]^\circ$ for all $x \in G$. Let $x \in G$. Since $1_H \cdot (\lambda_G(x^{-1})T) \in UC(\hat{G}) \cap VN_H(G) = [UC(\hat{H})]^\circ$, $1_{xH} \cdot T = \lambda_G(x)[1_H \cdot (\lambda_G(x^{-1})T)] \in \lambda_G(x)[UC(\hat{H})]^\circ$. □

Let X be a subspace of $VN(G)$. Let X_c denote the set $\{T \in X : \operatorname{supp} T$ is compact$\}$. X is said to satisfy *Condition (CD)* if X_c is norm dense in X. Observing the above proof, we can see that Proposition 3.5 holds for X whenever X has the following properties.

(1) $[X_H]^\circ = X \cap VN_H(G)$.
(2) $B(G) \cdot X \subseteq X$ (i.e., X is a $B(G)$-submodule of $VN(G)$).
(3) $\lambda_G(G)X \subseteq X$ (i.e., X is left translation invariant).
(4) X satisfies condition *(CD)*.

It is clear that $C^*_\delta(G)$, $C^*_\rho(G)$, and $UC(\hat{G})$ satisfy (1)–(4). Let X be one of the spaces $AP(\hat{G})$, $WAP(\hat{G})$, $AP_{_B}(\hat{G})$, and $WAP_{_B}(\hat{G})$. Then X satisfies (1) and (2). Note that $u \cdot (\lambda_G(x)T) = \lambda_G(x)({}_xu \cdot T)$ for $x \in G$, $u \in B(G)$, and $T \in VN(G)$. So, X also satisfies (3). Thus, $X = \overline{\text{span}}\{\lambda_G(G)[X_H]^\circ\}$ if (4) holds. In fact, we have

Theorem 3.6. *Suppose that X is one of the spaces $AP(\hat{G})$, $WAP(\hat{G})$, $AP_{_B}(\hat{G})$, and $WAP_{_B}(\hat{G})$. Let X_H denote the corresponding subspace of $VN(H)$. Then X satisfies (CD) if and only if [X_H satisfies (CD) and $X = \overline{span}\{\lambda_G(G)[X_H]^\circ\}$].*

Proof. Suppose X satisfies *(CD)*. By the above discussion, $X = \overline{\text{span}}\{\lambda_G(G)[X_H]^\circ\}$. Note that $X_c \cap VN_H(G) = [(X_H)_c]^\circ$, and $T \in X_c \Rightarrow 1_H \cdot T \in X_c \cap VN_H(G)$. Thus, by Corollary 3.4, $[X_H]^\circ = X \cap VN_H(G) = \overline{X_c} \cap VN_H(G) = \overline{X_c \cap VN_H(G)} = \overline{[(X_H)_c]^\circ} = [\overline{(X_H)_c}]^\circ$. Therefore, $X_H = \overline{(X_H)_c}$, i.e., X_H satisfies *(CD)*.

Conversely, suppose X_H satisfies *(CD)* and $X = \overline{\text{span}}\{\lambda_G(G)[X_H]^\circ\}$. Then $X = \overline{\text{span}}\{\lambda_G(G)[\overline{(X_H)_c}]^\circ\} = \overline{\text{span}}\{\lambda_G(G)\overline{[(X_H)_c]^\circ}\} \subseteq \overline{X_c}$, that is, X satisfies *(CD)*. □

Remark 3.7. Our discussion on when X is generated by the translates of elements in X_H is motivated by Chou [3, Lemma 4.3]. Unfortunately, there was a gap in the proof of this lemma. If H is *non-amenable* there (equivalently, if G is non-amenable), then $q_\alpha \in P(G) \setminus B_\rho(G)$ for each α. To get a norm convergent subnet of $(q_\alpha \cdot T)$, we need to assume that $T \in AP_{_B}(\hat{G})$. Therefore, the proof of [3, Lemma 4.3] actually showed that if H is an open normal subgroup of G such that G/H is amenable, then $AP_B(\hat{G}) = \overline{\text{span}}\{\lambda_G(G)[AP_B(\hat{H})]^\circ\}$. Due to this, Theorem 4.4 and Proposition 4.6 of [3] are true for $B(G)$-almost periodic and $B(G)$-weakly almost periodic functionals. We do not know whether they hold for $AP(\hat{G})$ or $WAP(\hat{G})$ when G is non-amenable.

4. Almost periodic functionals on $A(G)$

Let G be a locally compact group. Chou [3, Proposition 5.3] showed that $AP_B(\hat{G}) = AP(\hat{G})$ and $WAP_B(\hat{G}) = WAP(\hat{G})$ when G is amenable. We have the following slightly stronger assertion (see Remark 4.2).

Lemma 4.1. *If G is amenable, then $AP(\hat{G})$ and $WAP(\hat{G})$ satisfy (CD).*

Proof. Let G be an amenable locally compact group. Then there exists a bounded approximate identity (u_α) of $A(G)$. Clearly, we may assume that $\text{supp}(u_\alpha)$ is compact for each α. Let $T \in VN(G)$. Then $\text{supp}(u_\alpha \cdot T)$ is compact for each α and $u_\alpha \cdot T \to T$ in the w*-topology. If $T \in AP(\hat{G})$, then $u_\alpha \cdot T \to T$ in norm, and hence $T \in \overline{AP(\hat{G})_c}$. If $T \in WAP(\hat{G})$, then $u_\alpha \cdot T \to T$ weakly, and hence $T \in \overline{WAP(\hat{G})_c}^w = \overline{WAP(\hat{G})_c}$. □

Remark 4.2. Since $A(G)$ is a regular Banach algebra of functions on G and $A(G)$ is also an ideal in $B(G)$, it is not hard to see that (1) if $AP(\hat{G})$ satisfies *(CD)*, then $AP_B(\hat{G}) = AP(\hat{G})$; (2) if $WAP(\hat{G})$ satisfies *(CD)*, then $WAP_B(\hat{G}) = WAP(\hat{G})$. We do not know whether $AP(\hat{G})$ satisfies *(CD)* implies that G is amenable.

Lemma 4.3. *Let H be an open subgroup of G. Let X and X_H be the same spaces as in Theorem 3.6. If X_H is a C^*-algebra and $X = \overline{\text{span}}\{\lambda_G(G)[X_H]^\circ\}$, then X is also a C^*-algebra.*

Proof. Suppose X_H is a C^*-algebra and $X = \overline{\text{span}}\{\lambda_G(G)[X_H]^\circ\}$. Note that $\lambda_G(G)X \subseteq X$ and X is self-adjoint. Then $\lambda_G(G)X\lambda_G(G) \subseteq X$. Thus X is closed under multiplication and hence is a C^*-algebra. □

Combining Proposition 3.5, Theorem 3.6, Lemma 4.1, and Lemma 4.3, we have

Theorem 4.4. *Let G be a locally compact group and H an open amenable subgroup of G. Let X be one of the spaces $AP(\hat{G})$, $WAP(\hat{G})$, $AP_B(\hat{G})$, and $WAP_B(\hat{G})$, and let X_H denote the corresponding subspace of $VN(H)$. Then X satisfies (CD) if and only if $X = \overline{\text{span}}\{\lambda_G(G)[X_H]^\circ\}$.*

In particular, if G contains an open abelian subgroup, then

(a) X is a C^*-algebra if X satisfies (CD);

(b) G has the dual Bohr approximation property if and only if $AP(\hat{G})$ satisfies (CD).

When H is an open *normal* subgroup of G with G/H amenable, Chou showed that (1°) if $AP_B(\hat{H}) = C^*_\delta(H)$, then $AP_B(\hat{G}) = C^*_\delta(G)$; (2°) if $AP_B(\hat{H})$ (resp. $WAP_B(\hat{H})$) is a C^*-algebra, then so is $AP_B(\hat{G})$ (resp. $WAP_B(\hat{G})$) (see [3, Theorem 4.4 and Proposition 4.6] and Remark 3.7). In our Theorem 4.4 and the following corollary, we do not assume that H is a normal subgroup of G but we do need H to be amenable or abelian.

Corollary 4.5. *Let G be an amenable locally compact group and H an open subgroup of G. Then*

(a) $AP(\hat{G})$ (resp. $WAP(\hat{G})$) is a C^*-algebra iff so is $AP(\hat{H})$ (resp. $WAP(\hat{G})$);

(b) G has the dual Bohr approximation property iff so does H.

In particular, if G is amenable and contains an open abelian subgroup, then G has the dual Bohr approximation property and $WAP(\hat{G})$ is a C^*-algebra.

Remark 4.6. If G is an amenable locally compact group and contains an open abelian subgroup, then G_d is amenable (see Bédos [1] and Bekka et al. [2]). In this case, $AP(\hat{G}) = C^*_\delta(G)$ and $WAP(\hat{G})$ is a C^*-algebra (Corollary 4.5). Note that $C^*_\delta(G) \cong C^*_\rho(G_d) = AP(\hat{G_d})$ if G_d is amenable (see Bédos [1] and Granirer [7]). It is unknown in general whether $AP(\hat{G}) \cong AP(\hat{G_d})$ and $WAP(\hat{G})$ is a C^*-algebra when G_d is amenable.

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