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INJECTIVE RESOLUTIONS OF NOETHERIAN RINGS AND COGENERATORS

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Abstract. We give new construction of injective resolutions of complexes and bimodules. Applying this construction to an injective resolution of a Noetherian ring, we construct a Σ-embedding cogenerator for the category of modules of projective dimension $\leq n$. Moreover, for a Noetherian projective k-algebra R, we show that R satisfies the Auslander condition if and only if the flat dimension of every R-module M is equal to or larger than the one of the injective hull $\mathrm{E}(M)$.

0. Introduction

Let R be a ring, and let $0 \to R \to E^0 \to E^1 \to \dots$ be a minimal injective resolution of a left free module R of one generator. Bass studied injective resolutions of left Noetherian rings, in particular, of commutative Noetherian rings [B1], [B2]. In the case of commutative Noetherian rings, he showed that for a prime ideal P of R, the injective dimension of R_P is equal to the flat dimension of the injective hull $\mathrm{E}(R/P)$ of R/P [B2], [X]. He also showed that R is Gorenstein if and only if E^n is the direct sum of indecomposable injective R-modules $\mathrm{E}(R/P)$, where P are prime ideals of height n, for every $n \geq 0$ [B2]. In other words, for every $n \geq 0$, E^n is the direct sum of indecomposable injective R-modules I of which the flat dimensions are equal to n. Xu studied categorical properties of Gorenstein rings [X]. In the case of non-commutative Noetherian rings, Auslander provided the homological condition, which is called the Auslander condition, as one face of non-commutative versions of Gorenstein rings [FGR]. For a non-commutative Noetherian ring R satisfying the Auslander condition, Hoshino showed that every indecomposable injective left R-module I of flat dimension n is a direct summand of E^n [H1]. By using dualities of derived categories, we showed that if the injective dimensions of ${}_RR$ and R_R are finite, then every indecomposable injective left R-module appears in some E^n [Mi]. In this paper, by using a new construction of injective resolutions of complexes, we provide results concerning injective resolutions of Noetherian rings, projective dimension, flat dimension of R-modules. Moreover we study cogenerators for categories of modules of finite flat dimension in order to describe categorical properties of module categories.

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In Section 1, from the point of view of derived categories we give a new construction of injective resolutions of complexes (Lemma 1.1). And we define Σ-embedding cogenerators for additive categories, and study the relation between an injective resolution of a module U and an additive category arising from U (Propositions 1.4, 1.5). In Section 2, we show that $\bigoplus_{i=0}^{n} E^i$ is a Σ-embedding cogenerator for the category of left R-modules of projective dimension $\leq n$ (Theorem 2.2). As a corollary, we show that if the left injective dimension of R is finite, then $\bigoplus_{i\geq 0} E^i$ is a Σ-embedding cogenerator for the category of left R-modules of finite flat dimension (Corollary 2.3). Moreover, we get another proof of a result [Mi], Corollary 4.7 (Corollary 2.4). In Section 3, we study the case of a projective k-algebra R over a commutative ring k. For a bimodule ${}_A M_B$, we construct an injective resolution of M in $\mathsf{Mod}\, A \otimes_k B^{op}$ by using an injective resolution of M in $\mathsf{Mod}\, A$ (Theorem 3.5). As an application of this, we study injective resolutions of flat modules (Proposition 3.8), and we show that $\bigoplus_{i=0}^{n} E^i$ is an injective cogenerator for the category of left R-modules of flat dimension $\leq n$ (Theorem 3.9). In Section 4, we apply results of Section 3 to projective k-algebras satisfying the Auslander condition. As a consequence, we get the non-commutative version of categorical properties in commutative Gorenstein rings. In particular, in the case of a Noetherian projective k-algebra R over a commutative ring k, we show that R satisfies the Auslander condition if and only if the flat dimension of every R-module M is equal to or larger than the one of the injective hull $\mathrm{E}(M)$ (Theorem 4.1).

1. Injective resolutions of complexes

Throughout this paper, we assume that all rings have non-zero unity, and that all modules are unital. From the point of view of derived categories, we give the following "The piled resolution lemma". For a ring R, we denote by $\mathsf{Mod}\, R$ (resp., $\mathsf{mod}\, R$, $\mathsf{Inj}\, R$) the category of left (resp., finitely presented left, injective left) R-modules. Let $\mathcal{A}$ be an additive category, $\mathsf{C}(\mathcal{A})$ the category of complexes of objects in $\mathcal{A}$, $\mathsf{K}(\mathcal{A})$ the homotopy category of $\mathcal{A}$, and $\mathsf{K}^+(\mathcal{A})$, $\mathsf{K}^-(\mathcal{A})$ and $\mathsf{K}^b(\mathcal{A})$ the full subcategories of $\mathsf{K}(\mathcal{A})$ generated by bounded below complexes, bounded above complexes, bounded complexes, respectively. For an abelian category $\mathcal{A}$, let $\mathsf{D}^*(\mathcal{A})$ be the quotient category of $\mathsf{K}^*(\mathcal{A})$ by the multiplicative system of quasi-isomorphisms, where $* =$ nothing or $+$.

For a complex $X^\bullet := (X^i, d^i)$, we define the following truncation:

$$\tau_{\leq n} X^\bullet : \ldots \to X^{n-1} \to X^n \to 0 \to \ldots .$$

For a sequence $\{X_i^\bullet; f_i : X_{i+1}^\bullet \to X_i^\bullet\}_{i\geq 1}$ of complexes in $\mathsf{K}(\mathsf{Mod}\, R)$, we have the following distinguished triangle in $\mathsf{K}(\mathsf{Mod}\, R)$:

$$X^\bullet \to \prod\nolimits_i X_i^\bullet \xrightarrow{\text{1-shift}} \prod\nolimits_i X_i^\bullet \to .$$

We denote $X^\bullet$ by $\underleftarrow{\mathrm{hlim}} X_i^\bullet$, and call it the homotopy limit of the sequence. According to [BN], for a complex $X^\bullet \in \mathsf{K}(\mathsf{Mod}\, R)$, $X^\bullet$ is isomorphic to $\underleftarrow{\mathrm{hlim}} \tau_{\leq n} X^\bullet$ in $\mathsf{D}(\mathsf{Mod}\, R)$.

Lemma 1.1. *Let R be a ring, and $L^\bullet$ a complex $\ldots \to 0 \to L^0 \to L^1 \to L^2 \to \ldots$ in $\mathsf{K}^+(\mathsf{Mod}\, R)$. If $0 \to L^i \to I_i^0 \to I_i^1 \to \ldots$ is an injective resolution of L^i $(i \geq 0)$, then there is a quasi-isomorphism from $L^\bullet$ to a complex of the following form in*

$\mathsf{K}^+(\mathsf{Mod}\,R)$:

$$\ldots \to 0 \to I_0^0 \to \bigoplus_{i=0}^{1} I_i^{1-i} \to \bigoplus_{i=0}^{2} I_i^{2-i} \to \ldots \to \bigoplus_{i=0}^{n} I_i^{n-i} \to \ldots .$$

Proof. Let $I_i^\bullet$ be a complex $I_i^0 \to I_i^1 \to \ldots \to I_i^j \to \ldots$ $(i \geq 0)$; then L^i has a quasi-isomorphism to $I_i^\bullet$ in $\mathsf{D}^+(\mathsf{Mod}\,R)$. By the inductive step, we construct a complex $V_k^\bullet \in \mathsf{K}^+(\mathsf{Inj}\,R)$ which has a quasi-isomorphism $\tau_{\leq k}L^\bullet \to V_k^\bullet$ in $\mathsf{K}^+(\mathsf{Mod}\,R)$ as follows. First, we take $V_0^\bullet := I_0^\bullet$. Assume we have a complex $V_{k-1}^\bullet$ which satisfies the above condition. Since $I_k^\bullet[-k+1]$ belongs to $\mathsf{K}^+(\mathsf{Inj}\,R)$, the quasi-isomorphism $\tau_{\leq k-1}L^\bullet \to V_{k-1}^\bullet$ gives a bijection $\mathrm{Hom}_{\mathsf{K}^+(\mathrm{Mod}\,R)}(V_{k-1}^\bullet, I_k^\bullet[-k+1]) \to \mathrm{Hom}_{\mathsf{K}^+(\mathrm{Mod}\,R)}(\tau_{\leq k-1}L^\bullet, I_k^\bullet[-k+1])$. Then, for the quasi-isomorphism $L_k \to I_k^\bullet$ and a distinguished triangle $\tau_{\leq k}L^\bullet \to \tau_{\leq k-1}L^\bullet \to L^k[-k+1] \to$ in $\mathsf{K}^+(\mathsf{Mod}\,R)$, we have the following commutative diagram in $\mathsf{K}^+(\mathsf{Mod}\,R)$:

$$\begin{array}{ccc} \tau_{\leq k-1}L^\bullet & \longrightarrow & L^k[-k+1] \\ \downarrow & & \downarrow \\ V_{k-1}^\bullet & \longrightarrow & I_k^\bullet[-k+1] \end{array}$$

Therefore we can choose $V_{k-1}^\bullet \to I_k^\bullet[-k+1]$ as a map between complexes. By taking a mapping cone of $V_{k-1}^\bullet \to I_k^\bullet[-k+1]$, we have the following morphism between distinguished triangles in $\mathsf{K}^+(\mathsf{Mod}\,R)$:

$$\begin{array}{ccccccc} \tau_{\leq k}L^\bullet & \longrightarrow & \tau_{\leq k-1}L^\bullet & \longrightarrow & L^k[-k+1] & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ V_k^\bullet & \longrightarrow & V_{k-1}^\bullet & \longrightarrow & I_k^\bullet[-k+1] & \longrightarrow \end{array}$$

Since $V_k^\bullet$ is a mapping cone of $V_{k-1}^\bullet \to I_k^\bullet[-k+1]$, it is clear that $V_k^\bullet$ belongs to $\mathsf{K}^+(\mathsf{Inj}\,R)$. Since $\tau_{\leq k-1}L^\bullet \to V_{k-1}^\bullet$ and $L^k \to I_k^\bullet$ are quasi-isomorphisms in $\mathsf{K}^+(\mathsf{Mod}\,R)$, $\tau_{\leq k}L^\bullet \to V_k^\bullet$ is a quasi-isomorphism in $\mathsf{K}^+(\mathsf{Mod}\,R)$. We have a sequence of quasi-isomorphisms from $\{\tau_{\leq k}L^\bullet; \tau_{\leq k+1}L^\bullet \to \tau_{\leq k}L^\bullet\}$ to $\{V_k^\bullet; V_{k+1}^\bullet \to V_k^\bullet\}$, and then there exists a quasi-isomorphism $\alpha : \underleftarrow{\mathrm{hlim}}\tau_{\leq k}L^\bullet \to \underleftarrow{\mathrm{hlim}}V_k^\bullet$. On the other hand, by the above construction of $V_k^\bullet$, the morphism $\tau_{\leq k-1}V_k^\bullet \to \tau_{\leq k-1}V_{k-1}^\bullet$ is the identity in $\mathsf{C}(\mathsf{Mod}\,R)$ for all k, that is, the morphism $V_k^i \to V_{k-1}^i$ is the identity in $\mathrm{Mod}\,R$, for all $0 \leq i \leq k-1$. Therefore, there exists the inverse limit $\varprojlim V_k^\bullet$ which belongs to $\mathsf{K}^+(\mathsf{Inj}\,R)$. According to [BN, Remark 2.3], there exists a quasi-isomorphism $\beta : \varprojlim V_k^\bullet \to \underleftarrow{\mathrm{hlim}}V_k^\bullet$. Similarly, there exists a quasi-isomorphism $\gamma : L^\bullet \to \underleftarrow{\mathrm{hlim}}\tau_{\leq k}L^\bullet$. Since $\varprojlim V_k^\bullet$ belongs to $\mathsf{K}^+(\mathsf{Inj}\,R)$, there exists a right inverse $\tilde{\beta}$ of β in $\mathsf{K}^+(\mathsf{Mod}\,R)$. Put $\delta := \tilde{\beta}\alpha\gamma$; then we have the following commutative diagram in $\mathsf{K}^+(\mathsf{Mod}\,R)$:

$$\begin{array}{ccc} L^\bullet & \xrightarrow{\gamma} & \underleftarrow{\mathrm{hlim}}\tau_{\leq k}L^\bullet \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \varprojlim V_k^\bullet & \xrightarrow[\beta]{} & \underleftarrow{\mathrm{hlim}}V_k^\bullet \end{array}$$

Hence $L^\bullet \to \varprojlim V_k^\bullet$ is a quasi-isomorphism in $\mathsf{K}^+(\mathsf{Mod}\,R)$. By construction, $\varprojlim V_k^\bullet$ is the following form:

$$\dots \to 0 \to I_0^0 \to \bigoplus\nolimits_{i=0}^{1} I_i^{1-i} \to \bigoplus\nolimits_{i=0}^{2} I_i^{2-i} \to \dots \to \bigoplus\nolimits_{i=0}^{n} I_i^{n-i} \to \dots .$$

□

Remark 1.2. In the proof of Lemma 1.1, by direct calculation, it is not hard to see that the above morphism between distinguished triangles can be chosen to become a commutative diagram in $\mathsf{C}(\mathsf{Mod}\,R)$. Then we can directly construct a quasi-isomorphism $L^\bullet \to \varprojlim V_k^\bullet$.

Corollary 1.3. *Let R be a ring, and $0 \to Y_n \to \dots \to Y_1 \to Y_0 \to X \to 0$ an exact sequence of left R-modules. If $0 \to Y_i \to I_i^0 \to I_i^1 \to \dots$ is an injective resolution of Y_i $(0 \le i \le n)$, then X has the following injective resolution:*

$$0 \to X \to Q \to \bigoplus\nolimits_{i=0}^{n} I_i^{i+1} \to \bigoplus\nolimits_{i=0}^{n} I_i^{i+2} \to \dots ,$$

where Q is a direct summand of $\bigoplus_{i=0}^{n} I_i^i$.

Proof. By Lemma 1.1, the complex $\dots \to 0 \to Y_n \to \dots \to Y_1 \to Y_0 \to 0 \to \dots$ is isomorphic to the following complex in $\mathsf{D}^+(\mathsf{Mod}\,R)$:

$$V^\bullet : \dots \to 0 \to V^{-n} \to V^{-n+1} \to \dots \to V^{-1} \xrightarrow{d^{-1}} V^0 \xrightarrow{d^0} V^1 \to \dots ,$$

where V^k is isomorphic to $\bigoplus_{i=0}^{n+k} I_{i-k}^i$ if $0 > k \ge -n$, and is isomorphic to $\bigoplus_{i=0}^{n} I_i^{i+k}$ if $k \ge 0$. Since V^k is injective $(-n \le k \le -1)$ and $\mathrm{H}^i V^\bullet = O$ for $i \ne 0$, $\mathrm{Im}\, d^{-1}$ is injective. By $\mathrm{H}^0 V^\bullet \cong X$, $\mathrm{Ker}\, d^0$ is isomorphic to $X \oplus \mathrm{Im}\, d^{-1}$. Since the complex $V^0 \to V^1 \to \dots$ is an injective resolution of $X \oplus \mathrm{Im}\, d^{-1}$, we get the statement. □

For a left R-module U, we denote by $\mathsf{Add}\,U$ (resp., $\mathsf{add}\,U$) the category of left R-modules which are direct summands of direct sums of copies of U (resp., direct summands of finite direct sums of copies of U), and denote by $\mathsf{Res}^n(U)$ (resp., $\mathsf{res}^n(U)$) the full subcategory of $\mathsf{Mod}\,R$ (resp., $\mathsf{mod}\,R$) consisting of left R-modules X which have exact sequences $0 \to U_n \to \dots \to U_1 \to U_0 \to X \to 0$, where $U_i \in \mathsf{Add}\,U$ (resp., $\mathsf{add}\,U$) $(0 \le i \le n)$. We denote the injective dimension of ${}_RU$ (resp., the flat dimension of ${}_RU$) by $\mathrm{idim}_R\, U$ (resp., $\mathrm{fdim}_R\, U$). Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ a full subcategory of $\mathcal{A}$. We call an object $X \in \mathcal{A}$ a Σ-embedding cogenerator (resp., a finitely embedding cogenerator, a cogenerator) for $\mathcal{B}$ provided that every object in $\mathcal{B}$ admits an injection to some direct sum (resp., finite direct sum, direct product) of copies of X in $\mathcal{A}$.

Proposition 1.4. *Let R be a left coherent ring, U a finitely presented left R-module, and $0 \to U \to E^0 \to E^1 \to E^2 \to \dots$ an injective resolution of ${}_RU$. Then $\bigoplus_{i=0}^{n} E^i$ is a finitely embedding cogenerator for $\mathsf{res}^n(U)$.*

Proof. Given $X \in \mathsf{res}^n(U)$, we have the following exact sequence:

$$0 \to U_n \to \dots \to U_1 \to U_0 \to X \to 0,$$

where $U_i \in \mathsf{add}\,U$ $(n \ge i \ge 0)$. It is easy to see that every U_j has the following injective resolution:

$$0 \to U_j \to I_j^0 \to I_j^1 \to I_j^2 \to \dots ,$$

where $I_j^i \in \mathsf{add}(E^i)$ for all $i \ge 0$. By Corollary 1.3, we get an injection from X to $\bigoplus_{i=0}^{n} I_i^i$. Since $\bigoplus_{i=0}^{n} I_i^i$ belongs to $\mathsf{add}(\bigoplus_{i=0}^{n} E^i)$, we complete the proof. □

Proposition 1.5. *Let R be a left Noetherian ring, U a left R-module, and $0 \to U \to E^0 \to E^1 \to E^2 \to \ldots$ an injective resolution of ${}_RU$. Then $\bigoplus_{i=0}^n E^i$ is a Σ-embedding cogenerator for $\mathsf{Res}^n(U)$.*

Proof. The same as the proof of Proposition 1.4. ☐

2. Injective resolutions of rings and Σ-embedding cogenerators

In and after this section, we fix a minimal injective resolution of a left free module R of one generator:

$$0 \to {}_RR \to E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} E^2 \to \ldots .$$

Proposition 2.1. *Let R be a left coherent ring. Then $\bigoplus_{i=0}^n E^i$ is a finitely embedding cogenerator for the category of finitely presented left R-modules of projective dimension $\leq n$.*

Proof. By Proposition 1.4. ☐

Theorem 2.2. *Let R be a left Noetherian ring. Then $\bigoplus_{i=0}^n E^i$ is a Σ-embedding cogenerator for the category of left R-modules of projective dimension $\leq n$. In particular, every injective left R-module of projective dimension $\leq n$ is a direct summand of $\bigoplus_{i=0}^n E^i$.*

Proof. By Proposition 1.5. ☐

Corollary 2.3. *Let R be a left Noetherian ring. If $\mathrm{idm}({}_RR)$ is finite, then $\bigoplus_{i\geq 0} E^i$ is a Σ-embedding cogenerator for the category of left R-modules of finite flat dimension.*

Proof. According to [H2], Proposition 6, every left R-module of finite flat dimension is of finite projective dimension. Then we complete the proof by Theorem 2.2. ☐

We get another proof of a result [Mi], Corollary 4.7.

Corollary 2.4. ([Mi, Corollary 4.7]) *Let R be a right coherent and left Noetherian ring. If $\mathrm{idm}({}_RR)$ and $\mathrm{idm}(R_R)$ are at most n, then every indecomposable injective left R-module is a direct summand of $\bigoplus_{i=0}^n E^i$.*

Proof. According to [CE], Chap. VI, Proposition 5.6, for every finitely presented right R-module M and every injective left R-module I, we have the following isomorphism:

$$\begin{aligned} \mathrm{Tor}_i^R(M, I) &\cong \mathrm{Hom}_R(\mathrm{Ext}_R^i(M, R), I) \\ &= 0 \quad \text{for all} \quad i > n. \end{aligned}$$

Therefore, the flat dimension of every indecomposable injective left R-module is at most n. By Corollary 2.3, we get the statement. ☐

3. Injective resolutions of projective algebras and cogenerators

Let k be a commutative ring. We call a k-algebra R a projective k-algebra if R is projective as a k-module. Let A and B be projective k-algebras. According to [CE], a projective (resp., injective) $A \otimes_k B^{op}$-module is projective (resp., injective) as both a right B-module and a left A-module.

Lemma 3.1. *Let R be a left Noetherian ring, $\{Q_\lambda; f_{\mu,\lambda} : Q_\lambda \to Q_\mu\}_{\lambda\in\Lambda,\lambda\leq\mu}$ a direct system of injective left R-modules. Then $\varinjlim Q_\lambda$ belongs to $\mathsf{Add}(\bigoplus_{\lambda\in\Lambda} Q_\lambda)$.*

Proof. If $\varinjlim Q_\lambda = 0$, then there is nothing to prove. Otherwise, by [CE], Chap. I Ex. 8, and [Ma], $\varinjlim Q_\lambda$ is a direct sum $\bigoplus_{\gamma\in\Gamma} I_\gamma$ of indecomposable injective left R-modules I_γ ($\gamma \in \Gamma$). Given $\gamma \in \Gamma$, we choose a non-zero element x in $I_\gamma \subseteq \bigoplus_{\gamma\in\Gamma} I_\gamma$. Let $\Lambda(x)$ be the set of pairs (y, λ) such that $x = f_\lambda(y)$, where $y \in Q_\lambda$ for some $\lambda \in \Lambda$ and $f_\lambda : Q_\lambda \to \varinjlim Q_\lambda$ is the structure morphism. Since R is left Noetherian and Λ is a directed set, the set of left annihilator ideals $\mathrm{lann}(y)$ where $(y, \lambda) \in \Lambda(x)$ has a unique maximal left ideal, say, $\mathrm{lann}(y_0)$ where $y_0 \in Q_{\lambda_0}$. Then $\mathrm{lann}(x)$ is equal to $\mathrm{lann}(y_0)$. By [Ma], I_γ is a direct summand of Q_{λ_0}, and hence we complete the proof. □

Remark 3.2. Huisgen-Zimmermann and Smalø get a similar result of Lemma 3.1 in the case of Σ-pure injective modules (see [HS]).

Lemma 3.3. *Let k be a commutative ring, R a left Noetherian projective k-algebra, and $0 \to R \to V^0 \to V^1 \to V^2 \to \dots$ an injective resolution of R as left $R\otimes_k R^{op}$-modules. If F is a flat left R-module, then F has the following injective resolution:*

$$0 \to F \to I^0 \to I^1 \to I^2 \to \dots ,$$

where I^i belongs to $\mathsf{Add}({}_R V^i)$ for all $i \geq 0$.

Proof. Given a flat left R-module F, we have the following acyclic complex:

$$0 \to F \to V^0\otimes_R F \to V^1\otimes_R F \to V^2\otimes_R F \to \dots .$$

By [L], Theorem 1.2, there is a direct system $\{P_\lambda; f_{\mu,\lambda} : P_\lambda \to P_\mu\}_{\lambda\in\Lambda,\lambda\leq\mu}$ of finitely generated free left R-modules such that $\varinjlim P_\lambda$ is isomorphic to F. Then every $V^i\otimes_R F$ is isomorphic to $\varinjlim V^i\otimes_R P_\lambda$. By Lemma 3.1, $V^i\otimes_R F$ is an injective left R-module which belongs to $\mathsf{Add}({}_R V^i)$. Hence the above acyclic complex is an injective resolution of F which satisfies the statement. □

Proposition 3.4. *Let k be a commutative ring, R a left Noetherian projective k-algebra, and $0 \to R \to V^0 \to V^1 \to V^2 \to \dots$ an injective resolution of R as left $R\otimes_k R^{op}$-modules. Then $\bigoplus_{i=0}^n {}_R V^i$ is a Σ-embedding cogenerator for the category of left R-modules of flat dimension $\leq n$. In particular, every injective left R-module of flat dimension $\leq n$ is a direct summand of $\bigoplus_{i=0}^n {}_R V^i$.*

Proof. Let X be a left R-module of flat dimension $\leq n$; then X has the following flat resolution:

$$0 \to F_n \to \dots \to F_1 \to F_0 \to X \to 0.$$

By Lemma 3.3, every F_j has the following injective resolution:

$$0 \to F_j \to I_j^0 \to I_j^1 \to I_j^2 \to \dots ,$$

where $I_j^i \in \mathsf{Add}(V^i)$ for all $i \geq 0$. By Corollary 1.3, we get an injection from X to $\bigoplus_{i=0}^n I_i^i$. Since $\bigoplus_{i=0}^n I_i^i$ belongs to $\mathsf{Add}(\bigoplus_{i=0}^n {}_R V^i)$, we complete the proof. □

Let k be a commutative ring, A a projective k-algebra. We denote by $S_\bullet(A)$ the bar resolution of A, i.e. the complex $(\mathrm{S}_n(A), d_{n+1} : \mathrm{S}_{n+1}(A) \to \mathrm{S}_n(A))_{n\geq 0}$ such that $\mathrm{S}_n(A)$ is the $(n+2)$-fold tensor product over k of A with itself, and that $d_n(a_0\otimes \dots \otimes a_{n+1}) = \sum_{0\leq i\leq n+1} (-1)^i a_0\otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}$. In this case, $\mathrm{S}_\bullet(A)$ is a projective resolution of A in $\mathsf{Mod}\, A\otimes_k A^{op}$.

Theorem 3.5. *Let k be a commutative ring, A a k-algebra and B a projective k-algebra. For an $(A-B)$-bimodule M, let $0 \to M \to I^0 \to I^1 \to I^2 \to \dots$ be an injective resolution of M as left A-modules. Then M has the following injective resolution in $\mathsf{Mod}\, A \otimes_k B^{op}$:*

$$0 \to M \to V^0 \to V^1 \to V^2 \to \dots$$

such that $V^n = \bigoplus_{i=0}^{n} \mathrm{Hom}_k({}_B\,\mathrm{S}_{i-1}(B)_k, {}_A I_k^{n-i})$ for all $n \geq 0$, where $\mathrm{S}_{-1}(B) = B$.

Proof. It is easy to see that we have the following exact sequence in $\mathsf{Mod}\, A \otimes_k B^{op}$:

$$0 \to \mathrm{Hom}_B({}_B B_B, {}_A M_B) \to \mathrm{Hom}_B({}_B\,\mathrm{S}_0(B)_B, {}_A M_B) \to \mathrm{Hom}_B({}_B\,\mathrm{S}_1(B)_B, {}_A M_B) \to \dots .$$

Then ${}_A M_B$ is isomorphic to $\mathrm{Hom}_B({}_B\,\mathrm{S}_\bullet(B)_B, {}_A M_B)$ in $\mathsf{D}^+(\mathsf{Mod}\, A \otimes_k B^{op})$. By adjointness, for every $n \geq 0$, we have the following isomorphism in $\mathsf{Mod}\, A \otimes_k B^{op}$:

$$\mathrm{Hom}_B({}_B\,\mathrm{S}_n(B)_B, {}_A M_B) \cong \mathrm{Hom}_k({}_B\,\mathrm{S}_{n-1}(B)_k, {}_A M_k).$$

Since B is k-projective, by [CE], Chap. IX, Proposition 2.3a, $\mathrm{Hom}_k({}_B\mathrm{S}_{n-1}(B)_k, {}_A I_k^i)$ is injective as a left $A \otimes_k B^{op}$-module for every $n, i \geq 0$. Therefore, for every $n \geq 0$, $\mathrm{Hom}_k({}_B\,\mathrm{S}_{n-1}(B)_k, {}_A M_k)$ has the following injective resolution in $\mathsf{Mod}\, A \otimes_k B^{op}$:

$$0 \to \mathrm{Hom}_k({}_B\,\mathrm{S}_{n-1}(B)_k, {}_A M_k) \to \mathrm{Hom}_k({}_B\,\mathrm{S}_{n-1}(B)_k, {}_A I_k^0) \to \mathrm{Hom}_k({}_B\,\mathrm{S}_{n-1}(B)_k, {}_A I_k^1) \to \dots .$$

By Lemma 1.1, we complete the proof. □

Remark 3.6. In the proof of Theorem 3.5, we cannot induce an $(A-B)$-bimodule morphism $\mathrm{Hom}_k({}_B\,\mathrm{S}_{n-1}(B)_k, {}_A M_k) \to \mathrm{Hom}_k({}_B\,\mathrm{S}_n(B)_k, {}_A M_k)$ from any morphism of $\mathrm{Hom}_B({}_B\,\mathrm{S}_n(B)_k, {}_B\,\mathrm{S}_{n-1}(B)_k)$ in the case of non-commutative algebras. Then the above complex $V^\bullet$ cannot be described by a double complex in general.

For an injective left module I over a ring R, we denote by $\mathsf{Ps}(I)$ the category of left R-modules which are direct summands of direct products of copies of I.

Lemma 3.7. *Let R be a ring, I an injective left R-module.*

(*a*) *If R is a left Noetherian ring, then $\mathsf{Ps}(I)$ is closed under direct sums.*

(*b*) *If R is a right coherent ring and $\mathrm{fdim}_R\, I \leq n$, then every R-module of $\mathsf{Ps}(I)$ is of flat dimension $\leq n$.*

Proof. See [Ma] and [CE], Chap. II, Ex. 2. □

Proposition 3.8. *Let k be a commutative ring, and R a left Noetherian projective k-algebra. Then every flat left R-module F has the following injective resolution:*

$$0 \to F \to I^0 \to I^1 \to I^2 \to \dots ,$$

where $I^n \in \mathsf{Ps}(\bigoplus_{i=0}^{n} E^i)$ for all $n \geq 0$.

Proof. By Theorem 3.5, R has the following injective resolution in $R \otimes_k R^{op} - \mathsf{Mod}$:

$$0 \to R \to V^0 \to V^1 \to V^2 \to \dots ,$$

where $V^n = \bigoplus_{i=0}^{n} \mathrm{Hom}_k(\mathrm{S}_{i-1}(R), E^{n-i})$ for all $n \geq 0$. Since all $\mathrm{S}_{i-1}(R)$ are projective k-modules, it is easy to see that every V^n belongs to $\mathsf{Ps}(\bigoplus_{i=0}^{n} E^i)$. According to Lemmas 3.3 and 3.7, every flat R-module F has the following injective resolution:

$$0 \to F \to I^0 \to I^1 \to I^2 \to \dots ,$$

where $I^n \in \mathsf{Ps}(\bigoplus_{i=0}^{n} E^i)$ for all $n \geq 0$. Hence we get the statement. □

Theorem 3.9. *Let k be a commutative ring, and R a left Noetherian projective k-algebra. Then $\bigoplus_{i=0}^{n} E^i$ is an injective cogenerator for the category of left R-modules of flat dimension $\leq n$.*

Proof. Let X be a left R-module of flat dimension $\leq n$; then X has the following flat resolution :

$$0 \to F_n \to \ldots \to F_1 \to F_0 \to X \to 0.$$

By Proposition 3.8, every F_j has the following injective resolution:

$$0 \to F_j \to I_j^0 \to I_j^1 \to I_j^2 \to \ldots ,$$

where $I_j^m \in \mathsf{Ps}(\bigoplus_{i=0}^m E^i)$ for all $m \geq 0$. By Corollary 1.3, we get an injection from X to $\bigoplus_{i=0}^n I_i^i$. Since $\bigoplus_{i=0}^n I_i^i$ belongs to $\mathsf{Ps}(\bigoplus_{i=0}^n E^i)$, we complete the proof. □

Corollary 3.10. *Let k be a commutative ring, and R a Noetherian projective k-algebra which is a finitely generated k-module. Then $\bigoplus_{i=0}^n E^i$ is a Σ-embedding cogenerator for the category of left R-modules of flat dimension $\leq n$. In particular, every indecomposable injective left R-module of flat dimension $\leq n$ is a direct summand of $\bigoplus_{i=0}^n E^i$.*

Proof. Since R is a finitely generated k-module, we can replace $\mathsf{Ps}(\bigoplus_{i=0}^m E^i)$ by $\mathsf{Add}(\bigoplus_{i=0}^m E^i)$ in Proposition 3.8. Therefore we can also replace $\mathsf{Ps}(\bigoplus_{i=0}^m E^i)$ by $\mathsf{Add}(\bigoplus_{i=0}^m E^i)$ in the proof of Theorem 3.9. □

4. Applications to the Auslander condition

Let R be a coherent ring. Auslander defined the Auslander condition for R : for every finitely presented left R-module M and every $i \geq 0$, every submodule N of $\mathrm{Ext}_R^i(M, R)$ satisfies that $\mathrm{Ext}_R^j(N, R) = 0$ for all $0 \leq j < i$. This property implies the right side version of it. Moreover, the Auslander condition is equivalent to the following: $\mathrm{fdim}_R E^i \leq i$ for all $i \geq 0$ (see [FGR] for details). A Noetherian ring R is called Auslander-Gorenstein provided that R satisfies the Auslander condition and that $\mathrm{idim}(R_R)$ and $\mathrm{idm}({}_RR)$ are finite, and is called Auslander regular if R is an Auslander-Gorenstein ring of which global dimension is finite. For an R-module M, we denote by $\mathrm{E}(M)$ the injective hull of M.

Theorem 4.1. *Let k be a commutative ring, and R a right coherent and left Noetherian projective k-algebra. Then the following are equivalent.*
(a) *R satisfies the Auslander condition.*
(b) $\mathrm{fdim}_R \mathrm{E}(M) \leq \mathrm{fdim}_R M$ *for every left R-module M.*

Proof. (a) $\Rightarrow$ (b): We may assume $\mathrm{fdim}_R M = n < \infty$. According to Theorem 3.9, M admits an injection to some injective left R-module I in $\mathsf{Ps}(\bigoplus_{i=0}^n E^i)$. By Lemma 3.7, we have $\mathrm{fdim}_R I \leq n$. Since the injective hull $\mathrm{E}(M)$ of M is a direct summand of I, we have $\mathrm{fdim}_R \mathrm{E}(M) \leq n$.

(b) $\Rightarrow$ (a): If $\mathrm{fdim}_R \mathrm{Im}\, \delta^{n-1} \leq n$, then $\mathrm{fdim}_R E^{n-1} \leq n$ because of E^{n-1} being the injective hull of $\mathrm{Im}\, \delta^{n-1}$. Moreover, by dimension shift, we have $\mathrm{fdim}_R \mathrm{Im}\, \delta^n \leq n + 1$. Hence we complete the proof by induction on n. □

Proposition 4.2. *Let k be a commutative ring, and R a Noetherian projective k-algebra which satisfies the Auslander condition. Then $\bigoplus_{i=0}^n E^i$ is a Σ-embedding cogenerator for the category of left R-modules of flat dimension $\leq n$.*

Proof. By Theorem 3.9, it is sufficient to show that $\mathsf{Ps}(\bigoplus_{i=0}^n E^i) = \mathsf{Add}(\bigoplus_{i=0}^n E^i)$. Clearly, $\mathsf{Ps}(\bigoplus_{i=0}^n E^i)$ contains $\mathsf{Add}(\bigoplus_{i=0}^n E^i)$. Conversely, for every indecomposable injective left R-module I of $\mathsf{Ps}(\bigoplus_{i=0}^n E^i)$, the Auslander condition implies that

$\mathrm{fdim}_R\, I$ is at most n by Lemma 3.7. According to [H1], Theorem 6.3, I is a direct summand of $\bigoplus_{i=0}^{n} E^i$, and hence I belongs to $\mathsf{Add}(\bigoplus_{i=0}^{n} E^i)$. □

Proposition 4.3. *Let k be a commutative ring, and R an Auslander-Gorenstein projective k-algebra of injective dimension n. For a left R-module M, the following are equivalent.*

(a) $\mathrm{fdim}_R\, M \le m$.

(b) $\mathrm{idim}_R\, M \le n$ *and M has the following minimal injective resolution:*

$$0 \to M \to I^0 \to I^1 \to I^2 \to \dots ,$$

where $\mathrm{fdim}\,_R I^i \le min(i+m, n)$ *for all* $i \ge 0$.

Proof. $(a) \Rightarrow (b)$: Let $0 \to F_m \to \dots \to F_1 \to F_0 \to M \to 0$ be a flat resolution of M. According to [H2], Proposition 6, every flat left R-module F is of finite projective dimension. Then, by Corollary 1.3, it is easy to see $\mathrm{idim}_R\, F \le n$. By Proposition 3.8, it is easy to see that F_i has the following injective resolution:

$$0 \to F_i \to I_i^0 \to I_i^1 \to \dots \to I_i^n \to 0,$$

where $I^j \in \mathsf{Ps}(\bigoplus_{i=0}^{j} E^i)$ for all $j \ge 0$. Then by Corollary 1.3, M has the following injective resolution:

$$0 \to M \to Q^0 \to Q^1 \to \dots \to Q^n \to 0,$$

where Q^0 is a direct summand of $\bigoplus_{i=0}^{min(m,n)} I_i^i$, and $Q^j = \bigoplus_{i=0}^{min(m,n-j)} I_i^{i+j} (j \ge 1)$. Then, for every $0 \le j \le n$, we have $Q^j \in \mathsf{Ps}(\bigoplus_{i=0}^{min(j+m,n)} E^i)$. By the definition of Auslander-Gorenstein rings, we have $\mathrm{fdim}_R\, Q^j \le min(j+m, n)$ for all $j \ge 0$. Since every j-th term of a minimal injective resolution of M is a direct summand of Q^j, the condition (b) is satisfied.

$(b) \Rightarrow (a)$: Let $0 \to M \to I^0 \to I^1 \to I^2 \to \dots \to I^s \to 0$ be a minimal injective resolution of M, where $s \le n$. By dimension shifting, we have $\mathrm{fdim}_R\, M \le m$. □

Proposition 4.4. *Let k be a commutative ring, and R an Auslander-Gorenstein projective k-algebra of injective dimension $\le n+1$. Then we have $0 \le \mathrm{fdim}_R\, M - \mathrm{fdim}_R\, \mathrm{E}(M) \le n$ for every left R-module M of finite flat dimension.*

Proof. Let M be a left R-module of finite flat dimension. By [H2], Proposition 6, we have $\mathrm{fdim}_R\, M \le n+1$. According to Theorem 4.1, it suffices to consider the case of $\mathrm{fdim}_R\, M = n+1$. Let $M \to \mathrm{E}(M)$ be the injective hull of M; then we have the following short exact sequence:

$$0 \to M \to \mathrm{E}(M) \to C \to 0.$$

If $\mathrm{fdim}_R\, \mathrm{E}(M) = 0$, then $\mathrm{fdim}_R\, C = n+2$. By [H2], Proposition 6, this is contradiction. Therefore $\mathrm{fdim}_R\, \mathrm{E}(M) > 0$, and hence we complete the proof. □

Corollary 4.5. *Let k be a commutative ring, and R an Auslander regular projective k-algebra of injective dimension $\le n+1$. Then $0 \le \mathrm{fdim}_R\, M - \mathrm{fdim}_R\, \mathrm{E}(M) \le n$ for every left R-module M.*

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