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A CONTINUOUS DECOMPOSITION OF THE MENGER CURVE INTO PSEUDO-ARCS

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Abstract. It is proved that the Menger universal curve $\mathcal{M}$ admits a continuous decomposition into pseudo-arcs with the quotient space homeomorphic to $\mathcal{M}$.

Wilson proved [8] Anderson's announcement [1] saying that for any Peano continuum X the Menger universal curve $\mathcal{M}$ admits a continuous decomposition into homeomorphic copies of $\mathcal{M}$ such that the quotient space is homeomorphic to X. Anderson also announced (unpublished) that the plane admits a continuous decomposition into pseudo-arcs. This result was proved by Lewis and Walsh [4].

In a previous paper [6] the author has proved that each locally planar Peano continuum with no local separating point admits a continuous decomposition into pseudo-arcs. Applying this result, we prove in this note that the Menger universal curve $\mathcal{M}$ also admits such a decomposition. We can topologically obtain $\mathcal{M}$ and some other continua as the quotient space, but not all Peano continua.

General constructions and their properties

For any compact metric space X let $\Psi_\omega(X)$ be the set of all sequences $\{X_n\}$, for $n \in \mathcal{N} = \{\infty, \in, ...\}$, of closed, mutually disjoint, nonempty subsets of X. Next, let C be the standard Cantor set in the unit interval $[0,1]$. Fix a sequence of open intervals (a_n, b_n) composed of all, mutually different components of $[0,1]-C$. Given a compactum X and a sequence $\{X_n\} \in \Psi_\omega(X)$, in the product $C \times X$ identify all pairs of points $\langle a_n, x\rangle$ and $\langle b_n, x\rangle$, where $x \in X_n$ and $n \in \mathcal{N}$. Observe that this identification yields an upper semi-continuous decomposition of $C \times X$. Denote by $Q(X, \{X_n\})$ the quotient space of this decomposition and by q the quotient mapping.

Property 1. *For any compactum X and any sequence $\{X_n\} \in \Psi_\omega(X)$, we have $\dim X = \dim Q(X, \{X_n\})$.*

Proof. Letting $Q = Q(X, \{X_n\})$, observe that Q contains copies of X, and thus $\dim Q \geq \dim X$.

Fix any positive integer n, take the permutation $\{i_1, ..., i_n\}$ of $\{1, ..., n\}$ satisfying $0 < a_{i_1} < b_{i_1} < ... < a_{i_n} < b_{i_n} < 1$, and define $I_0 = [0, a_{i_1}], I_k = [b_{i_k}, a_{i_{k+1}}], I_n =$

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$[b_{i_n}, 1]$ for $k \in \{1, ..., n-1\}$. Consider the equivalence in Q identifying all pairs of points $q(\langle c_1, x\rangle)$, $q(\langle c_2, x\rangle)$, such that

$$\text{either } c_1, c_2 \in I_k \text{ for some } k \in \{0, ..., n\}, \text{ or}$$

$$c_1, c_2 \in I_k \cup I_{k+1} \text{ and } x \in X_{i_{k+1}} \text{ for some } k \in \{0, ..., n-1\}.$$

Evidently, this equivalence yields an upper semi-continuous decomposition of Q, so denote by $f_n : Q \to f_n(Q)$ the quotient mapping. We see that

$$f_n(Q) = f_n(q(\{a_1\} \times X)) \cup ... \cup f_n(q(\{a_n\} \times X)) \cup f_n(q(\{1\} \times X)).$$

Thus $f_n(Q)$ is composed of the union of $n+1$ topological copies of X. Therefore $\dim f_n(Q) = \dim X$. Since we have

$$\lim_n \max\{\operatorname{diam} f_n^{-1}(z) : z \in f_n(Q)\} = 0,$$

then $\dim Q \leq \dim X$. □

An easy proof of the following property is left to the reader.

Property 2. *If X is a continuum and $\{X_n\} \in \Psi_\omega(X)$, then $Q(X; \{X_n\})$ is a continuum.*

If, additionally, a point $y = q(\langle c, x\rangle)$ separates the continuum $Q(X; \{X_n\})$, then $c \in \{a_n, b_n\}$ and $X_n = \{x\}$ for some n.

Let X be a compactum, $\{X_n\} \in \Psi_\omega(X)$ and let $y = q(\langle c, x\rangle)$ be a point of $Q = Q(X; \{X_n\})$. Take any closed neighborhood K of x in X, and positive integers i, j such that $b_i < a_j$ and $q^{-1}(y) \subset (C \cap [b_i, a_j]) \times K$. Let n_k be a sequence of all positive integers such that $[a_{n_k}, b_{n_k}] \subset [b_i, a_j]$. Notice that the set $L = q((C \cap [b_i, a_j]) \times K)$ is a closed neighborhood of y in Q, and the family of all such sets L form a basis of closed neighborhoods of y in Q.

Assuming that X is a locally connected continuum, we can take K to be a continuum.

If, moreover, X_n converges to X in the sense of the Hausdorff distance, then $X_{n_k} \cap K \neq \emptyset$ for almost all k. If this intersection is empty for some k's, we modify interval $[b_i, a_j]$ to some interval $[b_l, a_m] \subset [b_i, a_j]$ so that we have again $q^{-1}(y) \subset (C \cap [b_l, a_m]) \times K$, and $(a_{n_k}, b_{n_k}) \cap [b_l, a_m] = \emptyset$ for those k's. Then for each r such that $[a_r, b_r] \subset [b_l, a_m]$ we have $X_r \cap K \neq \emptyset$. Therefore the neighborhood $J = q((C \cap [b_l, a_m]) \times K)$ of y in Q is connected. Indeed, J is homeomorphic to a space of the form $Q(K; \{X_{r_k} \cap K\})$, where r_k is a sequence of all positive integers r from the previous statement, so it is connected by Property 2.

If, additionally, each X_n is dense in itself, then $\{x\} \neq X_{r_k} \cap K$ for any k. Hence y cannot separate J by Property 2.

We have proved the following property.

Property 3. *If X is a locally connected continuum and a sequence $\{X_n\} \in \Psi_\omega(X)$ converges to X in the sense of the Hausdorff distance, then $Q(X, \{X_n\})$ is a locally connected continuum.*

If, additionally, each X_n is dense in itself, then $Q(X, \{X_n\})$ has no local separating point.

In the next proposition we provide a construction of mappings between the spaces of type $Q(X; \{X_n\})$. The proof of this proposition is easy and natural, so we omit it.

Proposition 4. *Let* $f : X \to Y$ *be a continuous mapping between compacta* X *and* Y*, and let* $\{Y_n\} \in \Psi_\omega(Y)$*. Then there is the unique mapping* $g : Q(X; \{f^{-1}(Y_n)\}) \to Q(Y; \{Y_n\})$ *such that (for* q_1*,* q_2 *being the respective natural quotient mappings) the diagram*

$$\begin{tikzcd}
C \times X \arrow[r, "\mathrm{id}_C \times f"] \arrow[d, "q_1"] & C \times Y \arrow[d, "q_2"] \\
Q(X; \{f^{-1}(Y_n)\}) \arrow[r, "g"] & Q(Y; \{Y_n\})
\end{tikzcd}$$

commutes, and this mapping is continuous.

Moreover, if f *is open (surjective), then* g *is open (surjective) too.*

Remark 5. Note that for the above mapping g and for any $p = q_2(\langle c, y\rangle)$ in $Q(Y; \{Y_n\})$ fiber $g^{-1}(p)$ is homeomorphic to fiber $f^{-1}(y)$.

Remark 6. Observe that, actually, the pattern of the Lebesgue mapping from the Cantor set C to an arc (identifying each pair a_n, b_n to a point) was employed in the construction of the spaces $Q(X; \{X_n\})$. Take any continuous surjection $m : C \to F$ such that

(i) F is a locally connected curve;
(ii) F contains a countable set F_0 and admits a basis $\{B_1, B_2, ...\}$ such that $\mathrm{bdB_i}$ is a finite subset of F_0 for each i; and
(iii) if $m^{-1}(p)$ is nondegenerate, then $p \in F_0$ for each $p \in F$.

For any compactum X and any sequence $\{X_n\} \in \Psi_\omega(X)$ we can obtain a space $Q_m(X; \{X_n\})$ analogous to the space $Q(X; \{X_n\})$, where mapping m plays the role of the Lebesgue mapping. These new spaces have properties similar to Properties 1, 2, 3 and to Proposition 4. Here we do not develop this generalization, for spaces $Q(X; \{X_n\})$ are sufficient for our purposes.

Decompositions of the Menger curve

First, we use the results of the previous section to obtain the following construction of topological Menger curves.

Proposition 7. *Let* X *be a locally connected curve containing no free arc. Then for any sequence* $\{X_n\} \in \Psi_\omega(X)$ *converging to* X*, such that each* X_n *is dense in itself, the space* $Q(X; \{X_n\})$ *is homeomorphic to the Menger universal curve* $\mathcal{M}$*.*

Proof. Applying Properties 1 and 2, we see that $Q = Q(X; \{X_n\})$ is a curve. Next, Q is locally connected and contains no local separating point by Property 3. Observe that the set $A = (C - \{a_1, b_1, a_2, b_2, \dots\}) \times X$ is dense in $C \times X$ and each of its open subsets contains an uncountable family of mutually exclusive simple triods. Further, the mapping $q : C \times X \to Q$ restricted to A is a homeomorphism. Therefore, each nonempty open subset of Q also contains uncountably many mutually disjoint simple triods. Hence such a subset cannot be embedded into the plane by the Moore triodic theorem [5].

Finally, applying the well-known Anderson characterization theorem for the Menger curve [2], we obtain the conclusion. ☐

Combining Propositions 7 and 4, the following general method of construction of upper semi-continuous (continuous) decompositions of the Menger curve is obtained.

Let X be a locally connected curve containing no free arc, and let $f : X \to Y$ be a continuous surjection such that each fiber $f^{-1}(y)$ is dense in itself. Next, take a sequence $\{Y_n\} \in \Psi_\omega(Y)$ such that sets $f^{-1}(Y_n)$ converge to X. Then mapping g of Proposition 4 induces an upper semi-continuous decomposition of the topological Menger curve $Q(X; \{f^{-1}(Y_n)\})$ (Proposition 7) into sets homeomorphic to the respective fibers of mapping f (Remark 5). Additionally,

(1) if f is an open mapping, then this decomposition is continuous (Proposition 4); and

(2) if Y is a curve containing no free arc and each Y_n is dense in itself, then the quotient space $Q(Y; \{Y_n\})$ of this decomposition is again a topological Menger curve (Proposition 7).

In a previous paper [6] the author has proved that the Sierpiński universal plane curve $\mathcal{S}$ admits an open mapping f onto itself with pseudo-arcs as all fibers. Let $X = Y = \mathcal{S}$ and take any sequence $\{Y_n\} \in \Psi_\omega(\mathcal{S})$ approximating $\mathcal{S}$ and composed of dense in themselves sets. Applying the above construction for this mapping f, we obtain the following main result of the paper.

Theorem 8. *There exists a continuous decomposition of the Menger universal curve $\mathcal{M}$ into pseudo-arcs such that the quotient space is homeomorphic to $\mathcal{M}$.*

Now, we briefly discuss the quotient spaces of continuous decompositions of $\mathcal{M}$ into pseudo-arcs. Let $\mathcal{C}_{\mathcal{M}}$ be the class of all such quotients. So $\mathcal{M} \in \mathcal{C}_{\mathcal{M}}$ by Theorem 8. Actually, an uncountable family of mutually non-homeomorphic, locally connected curves is contained in $\mathcal{C}_{\mathcal{M}}$. Indeed, they are obtained as $Q(\mathcal{S}; \{\mathcal{Y}\backslash\})$ when arbitrary compacta approximating $\mathcal{S}$ and admitting isolated points are substituted for Y_n in the last construction. In particular, if Z is the union of two copies M_1 and M_2 of the Menger curve such that the set $M_1 \cap M_2$ is embeddable into the plane and locally separates neither M_1, nor M_2 at each point, then Z can be obtained as such $Q(\mathcal{S}; \{\mathcal{Y}\backslash\})$. On the other hand, if all sets Y_n are finite, we see that $Q(\mathcal{S}; \{\mathcal{Y}\backslash\})$ is a curve containing no topological copy of $\mathcal{M}$. The generalizations mentioned in Remark 6 can also be used to produce some members of $\mathcal{C}_{\mathcal{M}}$.

However, not all locally connected continua belong to $\mathcal{C}_{\mathcal{M}}$. Indeed, applying [3], Th. 8, p.136, we see that each element of $\mathcal{C}_{\mathcal{M}}$ is one-dimensional. Moreover, we observe that for any $Z \in \mathcal{C}_{\mathcal{M}}$ each open subset of Z contains a simple closed curve. To see this notice that, otherwise, the pre-image of a dendrite in Z would be a subcontinuum of $\mathcal{M}$ with nonempty interior and trivial shape by [7], Th. 11, an impossibility. Thus we have the following.

Proposition 9. *Let $\mathcal{D}$ be a continuous decomposition of the Menger curve $\mathcal{M}$ into pseudo-arcs. Then the quotient space $\mathcal{M}/\mathcal{D}$ is a locally connected curve such that each nonempty, open subset of $\mathcal{M}/\mathcal{D}$ contains a simple closed curve.*

So, the question of characterization of spaces in $\mathcal{C}_{\mathcal{M}}$ naturally appears. In particular we have the following problem.

Problem 1. *Does there exist a locally connected curve $Z \notin \mathcal{C}_{\mathcal{M}}$ such that each open subset of Z contains a simple closed curve ? Does the Sierpiński curve $\mathcal{S}$ belong to $\mathcal{C}_{\mathcal{M}}$?*

In the previous paper the author characterized all locally planar Peano continua admitting continuous decomposition into pseudo-arcs as those without local separating points, and it was proved that any Peano continuum having a local separating point has no continuous decomposition into acyclic curves ([6], Th.16 and Pr.15). We end the paper with the following general problem.

Problem 2. *Characterize all Peano curves (continua) admitting continuous decomposition into pseudo-arcs (into acyclic curves).*

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