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CHARACTERISTIC CLASSES FOR COMPLEX BUNDLES WITH TRIVIAL REAL REDUCTION

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Abstract. This note concerns itself with a theory of characteristic classes for those complex bundles whose real reductions are trivial.

1

For a topological space X let $A_n(X)$ be the set of the complex n-bundles over X obtained by furnishing the trivial real bundle $X \times R^{2n} \to X$ a complex structure. Two bundles $\xi_0, \xi_1 \in A_n(X)$ are considered to be *equivalent* if there exists some $\xi \in A_n(X \times I)$ so that $\xi \mid X \times i = \xi_i$, $i = 0, 1$. Denote by $B_n(X)$ the set of all such equivalence classes.

It should be noted that our partition on $A_n(X)$ by the equivalence classes is subtler than the one given by isomorphism classes of complex bundles.

2

Elements of $B_n(X)$ can be classified within homotopy theory. Let CS_n be the space of all complex structures J's on the $2n$-dimensional Euclidean space R^{2n}. The operator

$$K : CS_n \times R^{2n} \to CS_n \times R^{2n}$$

defined by $K(J, v) = (J, Jv)$ equips the trivial real vector bundle $CS_n \times R^{2n} \to CS_n$ a complex structure. Denote by γ_n the resulting complex n-bundle over CS_n. It will be called *the canonical bundle* over CS_n.

For two topological spaces X, Y, let $[X, Y]$ be the set of homotopy classes of continuous maps $X \to Y$.

Proposition 1. *For any topological space X the correspondence $h : [X, CS_n] \to B_n(X)$, $h([f]) = f^*\gamma_n$, is a bijection.* □

Indeed the inverse of h is seen as follows. For a $\xi \in B_n(X)$ consider the complex structure on ξ as an R-linear morphism $J_\xi : X \times R^{2n} \to X \times R^{2n}$ of the real reduction of ξ. The map $f_\xi : X \to CS_n$ assigning to each $x \in X$ the complex structure $J_\xi \mid x \times R^{2n} \in CS_n$ is continuous and satisfies $f_\xi^*\gamma_n = \xi$. It will be called *the classifying map* of ξ.

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The space CS_n has two connected components, both diffeomorphic to the homogeneous space $SO(2n)/U(n)$. Denote by CS_n^+ the component that contains

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (n \text{ copies}),$$

and by CS_n^- the other.

Let $1 + c_1 + \cdots + c_n$ be the total Chern characteristic class for the restricted bundle $\gamma_n \mid CS_n^+$. We describe $H^*(CS_n^+; Z)$, the integral cohomology algebra of CS_n^+, in

Proposition 2. *The classes c_i, $i = 1, \cdots, n-1$, are all divisible by 2. Further if we put $e_i = \frac{1}{2}c_i$, then $e_1, \cdots, e_{n-1}$ form a simple system of generators for $H^*(CS_n^+; Z)$* [5, p.372] *and are subject to the relations*

$$R_i : e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \cdots + (-1)^i e_{2i} = 0, \quad i = 1, \cdots, n-1,$$

with the convention $e_k = 0$, $k > n-1$, being understood.

The cohomology algebra of the space CS_n^+ has been determined for Z_2 coefficients by C. Miller [3], and for Z_p (p a prime) coefficients by A. Borel [1]. The proof of Proposition 2 will be postponed until the final Section **7**.

By the first $[\frac{n+1}{2}] - 1$ relations each e_{2i} can be expressed as a polynomial g_i in e_{odd}. For instance the first four such polynomials are given by

$$\begin{aligned} g_1 &= e_1^2, \\ g_2 &= 2e_1e_3 - e_1^4, \\ g_3 &= e_3^2 + 2e_1e_5 - 4e_1^3e_3 + 2e_1^6, \\ g_4 &= 2e_3e_5 + 2e_1e_7 - 6e_1^2e_3^2 + 8e_1^5e_3 - 3e_1^8 - 4e_1^3e_5. \end{aligned}$$

Consequently substituting e_{2i} by g_i in the remaining $n - [\frac{n+1}{2}]$ relations gives rise to equations

$$h_k = 0, k = [\frac{n+1}{2}], \cdots, n-1,$$

where each h_k is a polynomial in e_{odd} of degree $4k$. For example when $n = 9$ these polynomials are

$$\begin{aligned} h_5 &= e_5^2 - 2g_2g_3 + 2e_3e_7 - 2g_1g_4, \\ h_6 &= g_3^2 - 2e_5e_7 + 2g_2g_4, \\ h_7 &= e_7^2 - 2g_3g_4, \\ h_8 &= g_4^2. \end{aligned}$$

Consider the algebra

$$\Phi = \begin{cases} Z[d_1, d_3, \cdots, d_{n-2}] \otimes \Lambda_Z(v_{2n+1}, v_{2n+5}, \cdots, v_{4n-5}) \text{ when } n \text{ is odd}, \\ Z[d_1, d_3, \cdots, d_{n-1}] \otimes \Lambda_Z(v_{2n-1}, v_{2n+3}, \cdots, v_{4n-5}) \text{ when } n \text{ is even}, \end{cases}$$

the tensor product of the polynomial algebra in d_{odd} with the exterior algebra in v_j. It is graded by

$$\dim(d_i) = 2i \text{ and } \dim(v_j) = j.$$

The differential $\delta : \Phi \to \Phi$ of degree 1 given by

$$\delta(d_i) = 0, \delta(v_j) = h_{[\frac{j+1}{4}]}(d_1, d_3, \cdots)$$

furnishes the algebra Φ with the structure of a differential graded commutative free algebra over Z. Moreover since CS_n^+ is a symmetric space, Proposition 2 implies

Proposition 3. *The algebra map* $l : (\Phi, \delta) \to (H^*(CS_n^+; Z), \delta = 0)$ *defined by* $l(d_i) = e_i, l(v_j) = 0$ *is the minimal model (over* Z*) for the space* CS_n^+ *(cf.* [2, p.158]*).*

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The connected component decomposition $CS_n = CS_n^+ \sqcup CS_n^-$ yields a natural partition $[X; CS_n] = [X, CS_n^+] \sqcup [X, CS_n^-]$. Let $B_n^{\pm}(X)$ be the h-image of $[X; CS_n^{\pm}]$ in $B_n(X)$. Since CS_n^+ and CS_n^- are mutually diffeomorphic, a theory of characteristic classes on $B_n^+(X)$ corresponds to one on $B_n^-(X)$. So our remaining discussion will focus on the former.

From now on assume that our space X is either a simply connected cell complex or has the rational homotopy type as a product of odd dimensional spheres. Let $(\Phi(X), \delta_X)$ be the rational minimal model of X. For a continuous map $f : X \to CS_n^+$, let $\Phi(f)$ be the homotopy class of a minimal model $\Phi \otimes Q \to \Phi(X)$ of f. Denote by $[\Phi \otimes Q, \Phi(X)]$ the set of homotopy classes of differential graded algebra maps $\Phi \otimes Q \to \Phi(X)$. Consider the functorial correspondence

$$[X, CS_n^+] \to [\Phi \otimes Q, \Phi(X)]$$

given by $[f] \to \Phi(f)$ [2, p.173].

Definition. Let $f : X \to CS_n^+$ be the classifying map of a $\xi \in B_n^+(X)$. The sets

$$d_i(\xi) = \{g(d_i \otimes 1) \in \Phi(X) \mid g \in \Phi(f)\}$$

will be called *the primary Chern characteristic sets* of ξ. The sets

$$v_j(\xi) = \{g(v_j \otimes 1) \in \Phi(X) \mid g \in \Phi(f)\}$$

will be called *the secondary Chern characteristic sets* of ξ. □

Since the forms d_i are closed in Φ, each set $d_i(\xi)$ consists of closed forms and the difference of any two such forms is a coboundary. Therefore passing to cohomology yields an unique class $\{d_i(\xi)\} \in H^{2i}(X; Q)$. The following corollary of Propositions 2 and 3 implies that these classes constitute nothing essentially new

Proposition 4. *In the rational cohomology* $H^*(X; Q)$ *the usual Chern characteristic classes* $c_1(\xi), \cdots, c_n(\xi)$ *of a* $\xi \in B_n^+(X)$ *can be given in terms of the primary Chern characteristic sets by the formulas*

$$c_{2k+1}(\xi) = 2\{d_{2k+1}(\xi)\};\ c_{2k}(\xi) = 2g_k(\{d_1(\xi)\}, \{d_3(\xi)\}, \cdots);\ c_n(\xi) = 0.$$ □

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It is the secondary Chern characteristic sets $v_j(\xi)$ that are of our interest. We observe that some homotopy and cohomology invariants can be extracted from the sets $v_j(\xi)$ even though the forms $v_j \otimes 1$ themselves are not closed in $\Phi \otimes Q$.

Observation 1. By rational homotopy theory the forms $v_j \otimes 1$ constitute a basis for $Hom(\pi_{odd}(CS_n^+), Q)$ and for a continuous map $f : X \to CS_n^+$, the set of induced chain maps

$$\Phi(f) : \Phi \otimes Q \to \Phi(X),$$

module decompositables, agrees with the dual action of the induced homotopy homomorphism

$$f_*: \pi_*(X) \to \pi_*(CS_n^+).$$

(See [2, p.175], or Lemma 3 in Section 6.) Thus the element

$$\pi_j(\xi) = v_j(\xi) \text{ module decompositables}$$

is well defined in $Hom(\pi_{odd}(X); Q) = \pi_{odd}(X) \otimes Q$.

Observation 2. Assume that our space X has the rational homotopy type as a product of odd-dimensional spheres. Then the minimal model $\Phi(X)$ agrees with the rational cohomology of X. Thus each set $v_j(\xi)$ actually consists of elements in $H^*(X; Q)$.

Observation 3. Suppose the bundle $\xi \in B_n(X)$ with classifying map f_ξ is such that

$$d_i(\xi) = \{0\}, i = 1, 3, \cdots$$

(this happens, in particular, when X is $2n-2$ connected). Let δ_X be the differential of $\Phi(X)$. Then

$$\begin{aligned} \delta_X \Phi(f_\xi)(v_j) &= \Phi(f_\xi)\delta v_j = \Phi(f_\xi) h_{\frac{j+1}{4}}(d_1, d_3, \cdots) \\ &= h_{\frac{j+1}{4}}(d_1(\xi), d_3(\xi), \cdots) = 0 \end{aligned}$$

indicates that $v_j(\xi)$ consists of closed forms. Similarly one can show that the difference of any two forms in $v_j(\xi)$ is a coboundary. That is, each set $v_j(\xi)$ survives to a unique element of $H^*(X; Q)$.

6

The homotopy and cohomology invariants associated with a $\xi \in B_n^+(X)$ in the previous section can well be nontrivial even if the usual Chern classes $c_1(\xi)$, $\cdots, c_{n-1}(\xi)$ all vanish.

Let $SO(2n)$ be the special orthogonal group of order $2n$, and let $f_k: SO(2n) \to CS_n^+$, for an integer $k \in Z$, be defined by $f_k(g) = g^k J_0 g'^k$, where $J_0 \in CS_n^+$ is specified in Section 3 and $'$ is the transpose operator. Denote by ξ_k the induced bundle $f_k^* \gamma_n$.

Theorem. *For the sequence of complex n-bundles $\{\xi_k \in B_n^+(SO(2n)) \mid k \in Z\}$*

1) *the secondary Chern characteristic classes π_i and v_i are distinctive;*
2) *the usual Chern characteristic classes c_i vanish.*

Using the grading $\bigoplus \Phi^r(X)$ of the model $\Phi(X)$ a graded vector space $I(X) = \bigoplus I^r(X)$ can be introduced by setting

$$I^r(X) = \Phi^r(X)/\text{decompositable forms}.$$

For instance it follows from Proposition 3 that

Lemma 1.

$$I(CS_n^+) = \begin{cases} span_Q\{e_1, e_3, \cdots, e_{n-1}, v_{2n-1}, v_{2n+3}, \cdots, v_{4n-5}\} & \text{if } n \text{ is even}, \\ span_Q\{e_1, e_3, \cdots, e_{n-2}, v_{2n+1}, v_{2n+5}, \cdots, v_{4n-5}\} & \text{if } n \text{ is odd}. \end{cases} \quad \square$$

For a compact connected Lie group G the minimal model $\Phi(G)$ agrees with $H^*(G;Q)$. From the well known isomorphisms

$$H^*(U(n);Q) = \Lambda_Q(y_1, y_3, \cdots, y_{2n-1}),$$
$$H^*(SO(2n);Q) = \Lambda_Q(x_5, x_7, \cdots, x_{4n-5}, w_{2n-1})$$

(where the generators y_i, x_i and w_{2n-1} are primitive with the suffixes indicating their dimensions) we get

Lemma 2.

$$I(U(n)) = span_Q\{y_1, y_3, \cdots, y_{2n-1}\},$$
$$I(SO(2n)) = span_Q\{x_3, x_7, \cdots, x_{4n-5}, w_{2n-1}\}. \quad \square$$

A continuous map $f : X \to Y$ induces a well defined homomorphism $I(f) : I(Y) \to I(X)$. We recall from [2, p.175] that

Lemma 3. *There is a canonical isomorphism $I^r(X) \to Hom(\pi_r(X), Q)$ which is natural with respect to homomorphisms induced by maps $X \to Y$.* $\square$

For an integer $k \in Z$ let l_k be the self-map of $SO(2n)$ defined by $g \to g^k$. Since the induced algebra map $H^*(SO(2n);Q) \to H^*(SO(2n);Q)$ is given by $l_k^*(x_i) = kx_i$, $l_k^*(w_{2n-1}) = kw_{2n-1}$, we have

Lemma 4. *The induced endomorphism $I(l_k)$ of $I(SO(2n))$ is multiplication by k.* $\square$

Clearly the map $f_1 : SO(2n) \to CS_n^+$ is the projection of the standard fibration

$$U(n) \subset SO(2n) \to CS_n^+.$$

Applying the natural transformation $Hom(\ ;Q)$ to the homotopy exact sequence of f_1 yields the exact sequence of vector spaces

$$\cdots \leftarrow I^r(U(n)) \leftarrow I^r(SO(2n)) \overset{I^r(f_1)}{\leftarrow} I^r(CS_n^+) \leftarrow I^{r-1}(U(n)) \leftarrow \cdots.$$

A dimension comparison discussion based on Lemmas 1 and 2 concludes

Lemma 5. *The homomorphism $I^r(f_1) : I^r(CS_n^+) \to I^r(SO(2n))$ is*
1) *an isomorphism for $r = 2n+1, 2n+5, \cdots, 4n-5$ when n is odd, and for $r = 2n+3, 2n+7, \cdots, 4n-5$ when n is even;*
2) *an injection for $r = 2n-1$ when n is even.* $\square$

Proof of the Theorem. Since $f_k = f_1 \circ l_k$, 1) is immediate from Lemmas 4 and 5.

For 2) consider the flag manifold $SO(2n)/T^n$ as the set of orthogonal decompositions of R^{2n} into oriented 2-planes $R^{2n} = L_1 \oplus \cdots \oplus L_n$. Assigning to each $L_1 \oplus \cdots \oplus L_n \in SO(2n)/T^n$ with $\omega_1 \oplus \cdots \oplus \omega_n \in CS_n^+$, where ω_i is the $\frac{\pi}{2}$ rotation on L_i with respect to the orientation, yields a map $g : SO(2n)/T^n \to CS_n^+$. In fact f_1 factors through the space $SO(2n)/T^n$ in the fashion

$$\begin{array}{ccc} SO(2n) & & \\ {\scriptstyle q}\downarrow & \searrow^{f_1} & \\ SO(2n)/T^n & \overset{g}{\to} & CS_n^+ \end{array}$$

where q is the standard projection (for $T^n \subset U(n)$). Since the induced bundle $g^*\gamma_n$ admits a canonical splitting into complex line bundles (this is indicated by our description of g) we find

$$g^*c_r(\gamma_n) = \text{the } r\text{th elementary symmetric polynomial in some } x_1, \cdots, x_n \in H^2(SO(2n)/T^n;Q).$$

The proof of 2) is now completed by the facts $H^2(SO(2n);Q)=0$ and $f_k = f_1 \circ l_k = g \circ q \circ l_k$. □

It also follows from Lemmas 2 and 5 that a sequence of complex n-bundles over each of the spheres

$$S^r, r = \begin{cases} 2n+1, 2n+5, \cdots, 4n-5 \text{ when } n \text{ is odd,} \\ 2n-1, 2n+3, \cdots, 4n-5 \text{ when } n \text{ is even,} \end{cases}$$

with distinctive secondary Chern classes $v_r \in H^r(S^r;Q) = Q$ (in the sense of Observation 3) can be obtained from the bundles ξ_k, $k \in Z$.

7

This section is devoted to a proof of Proposition 2.

Let $s_1, \cdots, s_{2n}$ be the standard vector space basis for R^{2n}, and let S^{2n-2} be the unit sphere in the subspace R^{2n-1} spanned by $s_1, \cdots, s_{2n-1}$. The map

$$\pi : CS_n^+ \to S^{2n-2}, \quad \pi(J) = Js_{2n} \in S^{2n-2}$$

is a fiber bundle projection whose fiber inclusion over $s_{2n-1} \in S^{2n-2}$, with respect to the standard splitting $R^{2n} = span\{s_1, \cdots, s_{2n-2}\} \oplus span\{s_{2n-1}, s_{2n}\}$, is

$$i_n : CS_{n-1}^+ \to CS_n^+, \quad i_n(J') = J' \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The proof proceeds by an induction on n. Let γ be the Hopf line bundle over the 2-sphere S^2. Since $CS_2^+ = S^2$ and since $\gamma_2 = \gamma \oplus \gamma$, Proposition 2 is true for $n = 2$. Assume that it has been proved for $n-1$.

Let $c_r(\gamma_n)$ be the rth Chern class of γ_n. Since $H^*(CS_n^+;Z)$ is torsion free and since the real reduction of γ_n is trivial, the 2-divisibility of $c_r(\gamma_n)$ follows from $c_r(\gamma_n) \equiv 0 \; mod\, 2$.

Consider $e_r(\gamma_n) = \frac{1}{2}c_r(\gamma_n)$, $r = 1, 2, \cdots, n-1$. By the naturality of Chern classes with respect to induced bundles we get from $i_n^*\gamma_n = \gamma_{n-1} \oplus \epsilon$ that

1) $i_n^*(e_r(\gamma_n)) = e_r(\gamma_{n-1})$, $r = 1, 2, \cdots, n-2$,

2) $i_n^*(e_{n-1}(\gamma_n)) = 0$,

where $i_n^* : H^*(CS_n^+;Z) \to H^*(CS_{n-1}^+;Z)$ is the induced homomorphism. Combined with the inductive hypothesis 1) indicates

Lemma 6. *The bundle π has Leray-Hirsch property* [5, p.365]. □

It now follows from 2) that, if we let $a \in H^{2n-2}(S^{2n-2};Z)$ be a generator, then

3) $e_{n-1}(\gamma_n) = k\pi^*a$ for some $k \in Z$.

We evaluate the integer k in

Lemma 7. $k = \pm 1$.

Proof. Let τ be the tangent bundle of the base S^{2n-2}. It has Euler class $\pm 2a$. The induced bundle $\pi^*\tau$ is a subbundle of γ_n whose orthogonal complement $(\pi^*\tau)^\perp$ over a $J \in CS_n^+$ is the 2-plane spanned by $s_{2n}, Js_{2n} \in R^{2n}$. Thus γ_n has a ready made decomposition into the Whitney sum of complex bundles $\pi^*\tau \oplus (\pi^*\tau)^\perp$ in which the second summand is trivial. This implies

$$c_{n-1}(\gamma_n) = c_{n-1}(\pi^*\tau).$$

However the top Chern class $c_{n-1}(\pi^*\tau)$ can be recognized as the Euler class of $\pi^*\tau$, which is $\pm 2\pi^* a$. □

The first statement of Proposition 2 has now been proved by Lemmas 6 and 7 (as well as our inductive hypothesis).

Finally since the real reduction of γ_n is trivial, the formulas expressing the Pontrjagin classes of γ_n in terms of its Chern classes [4, p.177] give rise to the equations

$$c_r(\gamma_n)^2 - 2c_{r-1}(\gamma_n)c_{r+1}(\gamma_n) + - \cdots \pm 2c_1(\gamma_n)c_{2r-1}(\gamma_n) \mp 2c_{2r}(\gamma_n) = 0.$$

Dividing both sides by the common divisor 4 yields the relations $R_1 - R_{n-1}$.

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