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A TOPOLOGICAL PROPERTY OF INTEGRABLE SYSTEMS

J. C. ÁLVAREZ PAIVA

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ABSTRACT. If we are given n real-valued smooth functions on $\mathbb{R}^{2n}$ which are in involution, then, under some mild hypotheses, the subset of $\mathbb{R}^{2n}$ where these functions are linearly independent is not simply connected.

A system of n smooth functions $f_i : \mathbb{R}^{2n} \to \mathbb{R}$, $1 \leq i \leq n$, is said to be in *involution* if the Poisson bracket of any two of them is zero. The classical topological property of systems in involution is given by the following theorem of Arnold:

Theorem ([1]). *Let $f_i : \mathbb{R}^{2n} \to \mathbb{R}$, $1 \leq i \leq n$, be n smooth functions in involution and let $c \in \mathbb{R}^n$ be a regular value of the moment map*

$$F := (f_1, \ldots, f_n) : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n \ .$$

A compact, connected component of $F^{-1}(c)$ is a Lagrangian torus in $\mathbb{R}^{2n}$.

In this note we show how a theorem of Viterbo (see [3]), which states that the Maslov class of an embedded Lagrangian torus in $\mathbb{R}^{2n}$ does not vanish, implies a second topological property for systems of functions in involution.

Theorem. *Let $f_i : \mathbb{R}^{2n} \to \mathbb{R}$, $1 \leq i \leq n$, be n smooth functions in involution. If for some regular value $c \in \mathbb{R}^n$ of the moment map $F : \mathbb{R}^{2n} \to \mathbb{R}^n$ the set $F^{-1}(c)$ contains a compact, connected component, then the set of points in $\mathbb{R}^{2n}$ where $df_1 \wedge df_2 \wedge \cdots \wedge df_n \neq 0$ is not simply connected.*

Example (The harmonic oscillator). Consider the Hamiltonian

$$H(q_1, q_2, p_1, p_2) := q_1^2 + q_2^2 + p_1^2 + p_2^2$$

for a harmonic oscillator. The two integrals of motion $f_1(q,p) := q_1^2 + p_1^2$ and $f_2(q,p) := q_2^2 + p_2^2$ Poisson commute and are independent everywhere except at the union of the two-dimensional planes $q_1 = p_1 = 0$ and $q_2 = p_2 = 0$. Using the radial projection onto the unit sphere, we see that the set where $df_1 \wedge df_2 \neq 0$ is homotopic to the 3-sphere with two fibers of the Hopf fibration deleted. We conclude that this set is not simply connected.

Proof of the theorem. By Arnold's theorem, a compact connected component of $F^{-1}(c)$ is an embedded Lagrangian torus. We will see that if it were possible to embed this torus into some open, simply connected set $V \subset \mathbb{R}^{2n}$ where F has no

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singular points, its Maslov class would vanish. This contradicts Viterbo's theorem ([3]).

Recall that the Maslov class of a Lagrangian submanifold of $\mathbb{R}^{2n}$ is defined as the pullback by the Gauss map of the generator μ of the first cohomology class of the Grassmannian of Lagrangian planes. If a Lagrangian torus in $F^{-1}(c)$ is contained in a set V without singular points, we may extend its Gauss map to the whole of V by assigning to a point $x \in V$ the Lagrangian subspace spanned by the Hamiltonian vector fields of the functions f_i , $i = 1, \ldots, n$, at the point x. Since V is simply connected, the pullback of the class μ to V under this map is trivial and so is its restriction to $F^{-1}(c)$. $\square$

Remark. This simple result and its proof were suggested by a theorem of Ehreshmann and Reeb ([2]) on the topology of the leaves of foliations in $\mathbb{R}^n$.

References

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Université Catholique de Louvain, Institut de Mathématique Pure et Appl., Chemin du Cyclotron 2, B–1348 Louvain–la–Neuve, Belgium

E-mail address: `alvarez@agel.ucl.ac.be`