

## COMPACT COMPOSITION OPERATORS ON THE SMIRNOV CLASS

JUN SOO CHOA, HONG OH KIM, AND JOEL H. SHAPIRO

(Communicated by Albert Baernstein II)

ABSTRACT. We show that a composition operator on the Smirnov class  $N^+$  is compact if and only if it is compact on some (equivalently: *every*) Hardy space  $H^p$  for  $0 < p < \infty$ . Along the way we show that for composition operators on  $N^+$  both the formally weaker notion of *boundedness*, and a formally stronger notion we call *metric compactness*, are equivalent to compactness.

### 1. INTRODUCTION

Every holomorphic self-map  $\varphi$  of open unit disc  $D$  of the complex plane induces, by means of the equation

$$C_\varphi f(z) = f(\varphi(z)) \quad (f \in H(D), z \in D),$$

a linear transformation  $C_\varphi$  called a *composition operator* on the vector space  $H(D)$  of all functions holomorphic on  $D$ . The true appeal of composition operators comes from Littlewood's Subordination Principle, which implies that each such operator takes every Hardy space  $H^p$  continuously into itself (see [3, Theorem 2.22, page 30], [4, Theorem 1.7, page 10], [16, Chapter 1]).

The  $H^p$ -continuity of composition operators raises the issue of how the function-theoretic properties of  $\varphi$  influence the operator-theoretic behavior of  $C_\varphi$  on various  $F$ -spaces (complete, metrizable topological vector spaces) of analytic functions. The past three decades have witnessed a flowering of research on this subject, much of which is summarized in two recent books: [3] and [16], with some of the most recent directions outlined in the conference proceedings [7].

The problem most immediately suggested by Littlewood's theorem, that of determining which composition operators are *compact* on the Hardy spaces, has been extensively studied, beginning with the 1969 dissertation of H. J. Schwartz [13]. A few years later Shapiro and Taylor [19] showed that the compactness problem for  $H^p$  does not depend on the index  $0 < p < \infty$ , and in 1987 Shapiro showed that it could be solved in terms of the asymptotic behavior of the Nevanlinna counting function of  $\varphi$  [15] (see also [16, Chapter 10]).

The goal of this paper is to show that the compactness problem's robustness with regard to the underlying Hardy space (meaning " $H^p$  space for  $0 < p < \infty$ ")

---

Received by the editors May 29, 1998 and, in revised form, September 10, 1998.

1991 *Mathematics Subject Classification*. Primary 47B38; Secondary 30D55.

*Key words and phrases*. Composition operator, Smirnov class, compact operator.

This research was supported in part by BSRI, KOSEF, and NSF.

persists even at the natural upper limit of these spaces: the *Smirnov class*  $N^+$ . Furthermore, we show that the apparently weaker property of *boundedness* (which is, for general  $F$ -spaces, *not* the same as continuity) is, for composition operators on  $N^+$ , actually *equivalent to compactness*. The chain of argument leads naturally to an apparently stronger property that we call “metric compactness,” which has come up in the context of the full Nevanlinna class  $N$ , and which we also show is equivalent, for composition operators on  $N^+$ , to compactness.

In the next section we carefully define the spaces  $N$  and  $N^+$ , as well as the accompanying notions of boundedness and compactness. Granting these definitions, here is the statement of our main result:

**1.1. Main theorem.** *For a holomorphic self-map  $\varphi$  of the unit disc, the following conditions are equivalent:*

- (a)  $C_\varphi$  is a bounded operator on  $N^+$ .
- (b)  $C_\varphi(N) \subset N^+$ .
- (c)  $C_\varphi$  is compact on  $H^p$  for some (equivalently: for all)  $0 < p < \infty$ .
- (d)  $C_\varphi$  is metrically compact on  $N$ .
- (e)  $C_\varphi$  is metrically compact on  $N^+$ .
- (f)  $C_\varphi$  is compact on  $N^+$ .

We prove this result in §4, after setting out the relevant definitions, prerequisites, and preliminary results in the next two sections.

**1.2. Remarks.** The equivalence (c)  $\leftrightarrow$  (d) was recently established by Choa and Kim [1]; the proof of our Main Theorem provides a more elementary argument.

About twenty years ago Nakamura and Yanagihara [10] showed that composition operators take  $N^+$  into itself, and characterized the compact ones in a manner that anticipated more sophisticated results of Sarason [12] on which our proof that (b)  $\leftrightarrow$  (c) is based. Nakamura and Yanagihara were the first to ask if the compactness problem for composition operators on  $N^+$  might be equivalent to the one for Hardy spaces. Earlier Yanagihara studied  $N^+$  as an  $F$ -space, showing that it is not locally convex, and identifying its dual space [22].

An abstract analogue of the Smirnov class, called the *Hardy Algebra*, plays an important role in the general theory of function algebras (see [5, Chapter V]).

## 2. PREREQUISITES

**2.1. Boundedness and compactness.** A linear transformation of a topological vector space  $X$  is called *compact* ([8, Chapter 5, Problem A, page 206]) if it takes some neighborhood of zero into a relatively compact set. This definition of compactness is equivalent to the usual one if  $X$  is a Banach space, and in 1954 J. H. Williamson showed that it renders the entire Riesz theory of compact operators valid for any topological vector space—even those that are not locally convex [21] (see also [8, Chapter 5, Problem B, page 207]).

Somewhat weaker than compactness is the notion of *boundedness*. A subset  $A$  of a topological vector space  $X$  is called *bounded* if, for every neighborhood  $V$  of the origin, there is a positive number  $t$  such that  $A \subset tV$  (see, for example, [8, page 44]). A *bounded linear transformation* on  $X$  is one that takes some neighborhood of zero to a bounded set. It is easy to check that relatively compact sets are bounded, hence compact linear transformations are bounded. Moreover, every bounded linear transformation is continuous, and conversely for any Banach space, or more

generally for any linear topological space possessing a bounded neighborhood of zero. But in spaces having *no* bounded neighborhood of zero (e.g. the space  $H(D)$  in its compact-open topology) the converse *fails*: the identity map is not bounded.

**2.2. Notation.** We employ the following simplifying notations:

- $m$  denotes Lebesgue measure on  $\partial D$ , normalized to have total mass one. Thus if  $f$  is a continuous complex-valued function on  $\partial D$ , then

$$\int_{\partial D} f \, dm = \frac{1}{2\pi} \int_{\partial D} f(e^{i\theta}) \, d\theta.$$

- $L^1$  denotes *real*  $L^1(m)$ ,  $M$  is the space of real-valued finite Borel measures on  $\partial D$ , and  $H^\infty$  is the space of bounded analytic functions on  $D$ .
- If  $f$  is a complex-valued function on  $D$  and  $0 \leq r < 1$ , then the *dilate of  $f$  by  $r$* ,  $f_r : \overline{D} \rightarrow \mathbb{C}$ , is defined by  $f_r(z) = f(rz)$  for  $|z| \leq 1$ .
- For  $x$  a positive number,  $\log^+ x = \max(\log x, 0)$ , so that  $\log x = \log^+ x - \log^+(1/x)$  for every  $x > 0$ .

**2.3. The Smirnov and Nevanlinna classes.** The *Nevanlinna class*  $N$  is the collection of functions  $f \in H(D)$  for which

$$\sup_{0 \leq r < 1} \int_{\partial D} \log^+ |f_r| \, dm < \infty.$$

Because of the numerical inequalities

$$(1) \quad \log^+ x \leq \log(1 + x) \leq \log^+ x + \log 2 \quad (x \geq 0)$$

the same definition results if the integrand  $\log^+ |f|$  is replaced by  $\log(1 + |f|)$ . We define

$$\|f\| \stackrel{\text{def}}{=} \sup_{0 \leq r < 1} \int_{\partial D} \log(1 + |f_r|) \, dm = \lim_{r \rightarrow 1^-} \int_{\partial D} \log(1 + |f_r|) \, dm$$

(the last equality here reflects the subharmonicity of  $\log(1 + |f|)$ , which causes its integral means to increase with  $r$ ), and set  $d(f, g) = \|f - g\|$  for each pair  $f, g \in N$ . The functional  $d$  thus defined is a natural translation-invariant metric that makes the vector space  $N$  into an additive topological group. However  $N$  is *not* a topological vector space—its scalar multiplication is discontinuous (see [17], and §3.3 below).

$N$  contains each Hardy space, and just as for Hardy spaces, for each  $f \in N$  the *radial limit*

$$f^*(e^{i\theta}) \stackrel{\text{def}}{=} \lim_{r \rightarrow 1^-} f(re^{i\theta})$$

exists for a.e. real  $\theta$  [4, Theorem 2.2, page 17]. The Hardy spaces actually lie in the *Smirnov Class*  $N^+$ , which is the subspace of  $N$  consisting of functions  $f$  for which

$$\int_{\partial D} \log^+ |f^*| \, dm = \lim_{r \rightarrow 1^-} \int_{\partial D} \log^+ |f_r| \, dm,$$

or, equivalently, for which the family of functions  $\{\log^+ |f_r| : 0 \leq r < 1\}$  is uniformly integrable on the unit circle. Thanks once again to inequalities (1) it makes no difference in this definition if  $\log^+ |f|$  is replaced by  $\log(1 + |f|)$ . Because of this

and the Vitali Convergence Theorem [11, Chapter 6, page 133, Problem 10] we have:

$$\|f\| = \int_{\partial D} \log(1 + |f^*|) dm \quad (f \in N^+),$$

from which the Dominated Convergence Theorem can be used to show that scalar multiplication is continuous on  $N^+$ . Thus  $N^+$  is a metrizable topological vector space, and standard arguments show that its metric is complete, hence it is an  $F$ -space (see [17, page 919] for details).

More is true:  $N^+$  is an algebra under pointwise multiplication. This follows immediately from the characterization of  $N^+$  in terms of uniform integrability, and the subadditivity of  $\log^+$ :

$$(2) \quad \log^+ xy \leq \log^+ x + \log^+ y \quad (x, y \geq 0).$$

A closer analysis shows that  $N^+$  is a *topological algebra*, i.e., pointwise multiplication is a continuous mapping  $N^+ \times N^+ \rightarrow N^+$ . This is an easy consequence of Vitali's Convergence Theorem and the following useful convergence criterion, which itself follows immediately from inequality (1).

**2.4. Lemma.** *Suppose  $\{f_n\}$  is a sequence in  $N^+$  and  $f \in N^+$ . Then  $f_n \rightarrow f$  in  $N^+$  if and only if on  $\partial D$ :  $f_n^* \rightarrow f^*$  in measure and  $\{\log^+ |f_n^*|\}$  is uniformly integrable.*

For more on uniform integrability see [5, Chapter V].

**2.5. Factorization in the Nevanlinna class.** We require some standard factorization theorems, dating back to 1929 work of Smirnov [20], for functions in the Nevanlinna class. Suppose  $f \in N$ . Then Jensen's formula guarantees that the zeros  $\{z_n\}$  of  $f$  (counted according to multiplicity) satisfy the Blaschke condition  $\sum_n (1 - |z_n|) < \infty$  (see [11, Theorems 15.18 and 15.23]). Thus  $f = BG$  where  $B$  is the Blaschke product formed from these zeros, and  $G$  is holomorphic and never-vanishing on  $D$ . Since  $\lim_{r \rightarrow 1^-} \int_{\partial D} \log |B_r| dm = 0$  [11, Theorem 15.24], it follows easily that  $G \in N$  and  $\|G\| = \|f\|$ . This decomposition  $f = BG$  is our *first fundamental factorization* of functions in  $N$ .

In this factorization  $G$  vanishes nowhere on  $D$ , so  $\log |G|$  is harmonic on  $D$ . Now  $\sup_{0 \leq r < 1} \int_{\partial D} \log^+ |G_r| dm < \infty$  because  $G \in N$ , so in particular if  $r_k \rightarrow 1^-$ , then the measures  $\log^+ |G_{r_k}| dm$  have a cluster point  $\mu^+$  in the weak star topology of  $M$ . It is not difficult to show that  $P[\mu^+]$  is the *least harmonic majorant* of  $\log^+ |G|$  (see, e.g., [6, Theorem 5.3, page 70]), hence  $\mu^+$  is independent of the sequence  $\{r_k\}$ . Thus  $\lim_{r \rightarrow 1^-} \log^+ |G_r| dm = \mu^+$  in the weak star topology.

Now the mean value property of harmonic functions (and the fact that  $\log |G(0)| > -\infty$ ) insures that the conclusions of the last paragraph still hold when  $\log^+ |G|$  is replaced by  $\log^+(1/|G|)$ . Thus there is another finite positive Borel measure  $\mu^-$  on  $\partial D$  for which  $\lim_{r \rightarrow 1^-} \log^+(1/|G_r|) dm = \mu^-$  (weak star limit). These observations lead to the following additional weak star limits:

$$\lim_{r \rightarrow 1^-} \log |G_r| dm = d\mu^+ - d\mu^- \quad \text{and} \quad \lim_{r \rightarrow 1^-} |\log |G_r|| dm = d\mu^+ + d\mu^-.$$

Thus, associated with each  $f \in N$  there is the non-vanishing function  $G \in N$  defined by  $G = f/B$ , and the real Borel measure  $\mu = \mu^+ - \mu^-$  for which  $\log |G| = P[\mu]$  (Poisson integral) on  $D$ . Upon holomorphically completing the Poisson integrals of  $\mu^+$  and  $\mu^-$ , and then exponentiating the result, we obtain our

*second fundamental factorization:*  $f = g/h$ , where  $g$  and  $h$  belong to  $H^\infty$ , both have moduli bounded by 1 on  $D$ , and  $h$  vanishes at no point of  $D$ .

The third factorization of  $f$  comes from decomposing  $\mu$  into singular and absolutely continuous parts. It turns out that the absolutely continuous part of  $\mu$  is just  $\log |G^*|$ , where  $G^*$  is the radial limit function of  $G$  (see once again [6, Theorem 5.3]), so  $d\mu = \log |G^*| dm + d\sigma$  where  $\sigma$  is singular with respect to  $m$ . Now  $\sigma = \delta - \nu$ , where  $\delta$  and  $\nu$  are positive finite Borel measures on  $\partial D$  that are mutually singular (as well as being singular with respect to  $m$ ). The measure  $\delta$  is particularly important in our development. Following the terminology in [17] we call it the *denominator measure* of  $f$ .

To produce the desired factorization of  $f$ , let  $\gamma[\delta]$  denote the exponential of the holomorphic completion of the Poisson integral of  $\delta$ . Employing similar notation for the other two measures discussed in this paragraph we have

$$(3) \quad f = \frac{\omega \cdot B \cdot \gamma[\log |G^*| dm] \cdot \gamma[-\nu]}{\gamma[-\delta]},$$

where  $B$  is the Blaschke product formed with the zeros of  $f$  and  $\omega$  is a complex number of modulus one.

Observe that the numerator on the right-hand side of (3), being a constant multiple of the product of three  $N^+$ -functions, also belongs to  $N^+$ , while the denominator has the form

$$\gamma[-\delta] = \exp \left\{ \int_{\partial D} \frac{z + \zeta}{z - \zeta} d\delta(\zeta) \right\} \quad (z \in D).$$

The usual notation for this function is  $S_\delta$ ; it is a bounded analytic function on  $D$ , with modulus bounded by 1 there, and has radial limits of modulus 1 at almost every point of  $\partial D$ . In other words, like  $B$ , it is an *inner function* [4, §2.4, page 24] (of course  $S_\delta$  differs from  $B$  in that it vanishes at no point of  $D$ ). Note that in the trivial case  $\delta \equiv 0$  we have  $S_\delta \equiv 1$ . Similar remarks apply, of course, to  $\gamma[-\nu] = S_\nu$ . The remaining factor  $H = \gamma[\log |G^*| dm]$  on the right-hand side of (3) is called the *outer factor* of  $f$ . This function belongs to  $N^+$ , vanishes nowhere, and  $|G^*| = |f^*|$  a.e. on  $\partial D$ . Thus  $f = \omega B S_\nu H / S_\delta$ , where the numerator is in  $N^+$  and  $S_\nu$  and  $S_\delta$  have no common inner factor. This is our *third fundamental factorization* of  $f$ . It turns out that this factorization into the product of unimodular constant, a Blaschke product, an outer function, and a quotient of relatively prime singular inner functions is unique (see [11, §17.14–17.19], [4, Theorem 2.9, page 25], or [6, Theorem 5.5, page 74] for more details).

The singular measure  $\delta$  in this last factorization is particularly important: if it is not the zero-measure, then it acts as an obstruction preventing  $f$  from belonging to  $N^+$ . Indeed, it is easy to check that the family of functions  $\{\log^+ |f_r| : 0 \leq r < 1\}$  is uniformly integrable on  $\partial D$  if and only if  $\delta = 0$  (this story continues in §3.2 and §3.3). In particular, a function in  $N^+$  is *invertible* if and only if it is *outer*.

It follows immediately from the definition given above that, for every outer function  $H$ ,

$$(4) \quad \log |H(0)| = \int_{\partial D} \log |H^*| dm.$$

It is not difficult to see from the Blaschke-singular-outer factorization of functions in  $N^+$  that this extension of the mean value property to the boundary *characterizes* the outer ones.

**2.6. Composition operators on  $N$  and  $N^+$ .** The fact that every composition operator maps  $N$  continuously into itself is most easily seen by recalling that  $f \in N$  if and only if the subharmonic function  $\log(1 + |f|)$  has a harmonic majorant (see [11, Exercise 4, page 352] for the corresponding result with  $\log^+ |f|$  in place of  $\log(1 + |f|)$ —the proof in our case is the same). Moreover if  $f \in N$ , then the *least* harmonic majorant  $\Lambda$  of  $\log(1 + |f|)$  is the Poisson integral of the measure that is the weak star limit of the measures  $\log(1 + |f_r|) dm$  as  $r \rightarrow 1-$ . In particular,  $\|f\| = \Lambda(0)$ .

Now if  $\varphi$  is a holomorphic self-map of  $D$ , then  $\Lambda \circ \varphi$  is a harmonic majorant for  $\log(1 + |f \circ \varphi|)$ , hence  $f \circ \varphi \in N$ , and

$$\|f \circ \varphi\| \leq \Lambda(\varphi(0)) \leq \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \Lambda(0) = \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \|f\|,$$

where the second inequality is an application of Harnack's inequality. This shows that  $C_\varphi$  acts continuously on  $N$ .

To show that every composition operator takes  $N^+$  into itself, we need only observe that  $N^+$  is the closure in  $N$  of  $H^\infty$ . The key here is that the Vitali Convergence Theorem guarantees that for each  $f \in N^+$  the dilates  $f_r$  defined by  $f_r(z) = f(rz)$  converge in  $N$  to  $f$  as  $r \rightarrow 1-$ . Since each of these dilates is bounded on  $D$ , this shows that  $H^\infty$  is dense in  $N^+$ . But composition operators preserve  $H^\infty$ , so their  $N$ -continuity insures that they also preserve  $N^+$ .

**2.7. The harmonic Hardy space  $h^1$ .** This is the space of real-valued harmonic functions  $u$  on  $D$  for which

$$\|u\|_1 \stackrel{\text{def}}{=} \sup_{0 \leq r < 1} \int_{\partial D} |u_r| dm < \infty.$$

Each  $u \in h^1$  is the Poisson integral of a finite real Borel measure  $\mu$  on  $\partial D$ , and the correspondence  $u \rightarrow \mu$  is a linear isometry taking  $h^1$  onto  $M$ , the space of all such measures on  $\partial D$ , regarded as a Banach space in its variation norm. In particular,  $h^1$  is a Banach space. We denote by  $h_0^1$  the subspace of  $h^1$  consisting of Poisson integrals of measures that are absolutely continuous with respect to Lebesgue measure. Thus  $h_0^1$  is isometrically isomorphic with  $L^1(\partial D)$ , and is therefore a closed subspace of  $h^1$ .  $h_0^1$  can be characterized as those  $u \in h^1$  for which the family of functions  $\{|u_r| : 0 \leq r < 1\}$  is uniformly integrable on  $\partial D$ .

The work of §2.5 reveals a close connection between  $h^1$  and  $N$ . Suppose  $f \in N$  and write  $f = BG$ , where  $B$  is the Blaschke product formed with the zeros of  $f$ . In §2.5 we noted that  $\log |G| \in h^1$ , and the remarks of the last paragraph show that  $f \in N^+$  if and only if  $\log^+ |G| \in h_0^1$ . Furthermore,  $h^1$  can be characterized as the collection of real-valued harmonic functions on  $D$  whose absolute values have harmonic majorants. Once this is done, arguments similar to those of §2.6 show that every composition operator maps  $h^1$  continuously into itself, and preserves  $h_0^1$ .

In [12] Sarason proved that a composition operator is compact on  $h^1$  if and only if it is compact on  $h_0^1$ , and that this happens if and only if it maps  $h^1$  into  $h_0^1$ . Now for composition operators, compactness on  $h^1$  implies compactness on the (holomorphic) Hardy space  $H^1$ , and therefore on every  $H^p$  for  $0 < p < \infty$ . Conversely, in [18], Shapiro and Sundberg proved that, if a composition operator is compact on a Hardy space, then it is compact on  $h^1$ . Thus for each holomorphic self map  $\varphi$  of  $D$ :  $C_\varphi$  is compact on a Hardy space  $\iff C_\varphi(h^1) \subset h_0^1$ .

Alec Matheson has pointed out that from Sarason’s results and the third fundamental factorization of §2.5 one can derive an alternate proof that composition operators preserve  $N^+$ . Indeed, because composition operators preserve  $h_0^1$  they also preserve outer functions, and the desired result now follows from the characterization of  $N^+$  as quotients of bounded functions, where the denominator is outer.

We remark in passing that in [12] Sarason asked how one might derive the result of Shapiro and Sundberg using purely function theoretic means, and this motivated subsequent work of Cima and Matheson, and of Jonathan Shapiro (see [2] and [14]). The earlier paper [10] of Nakamura and Yanagihara can be regarded as a forerunner of this entire body of work.

**2.8. Metric compactness.** Our work leads naturally to a somewhat stronger concept of operator compactness that was introduced for the Nevanlinna class by Masri [9]. We call an operator on  $N$  (or, equally well, on  $N^+$ ) *metrically compact* if it takes *every metric ball* into a relatively compact set (recall that for  $N^+$  the usual definition of compactness requires only that the image of *some* ball be relatively compact). Relative to a bounded metric only the zero-operator is metrically compact, however for composition operators on  $N$  our Main Theorem shows the metrically compact ones to be just those that restrict compactly to  $N^+$  (hence to  $H^2$ ).

### 3. PRELIMINARY RESULTS

We use the symbol  $\kappa$  for the compact-open topology on  $H(D)$ , and write  $f_\alpha \xrightarrow{\kappa} f$  for convergence in this topology (uniform convergence on compact subsets of  $D$ ) of a net  $(f_\alpha)$  to  $f$ . We write  $B_R$  for the closed ball in  $N$  of radius  $R > 0$ , centered at the origin, and  $B_R^+$  for the corresponding ball in  $N^+$ ; that is:  $B_R = \{f \in N : \|f\| \leq R\}$  and  $B_R^+ = B_R \cap N^+$ .

**3.1. Lemma.**  $B_R$  is  $\kappa$ -compact in  $N$  for every  $R > 0$ .

The proof is a consequence of Montel’s theorem and the fact that each function in the Nevanlinna class obeys a growth condition that can be expressed in terms of its distance in  $N$  to the origin [17, Proposition 1.1, page 917]. It is a routine extension of arguments familiar to all Hardy space enthusiasts, so we omit it.

The next result shows that the denominator measure  $\delta$  in the Third Fundamental Factorization of §2.5 plays an important role in the computation of distances in  $N$ .

**3.2. Lemma.** Suppose  $f \in N$  and  $\delta$  is its denominator measure, so that  $f = F/S_\delta$  where  $F \in N^+$  and  $S_\delta$  is relatively prime to the inner factor of  $F$ . Then

$$\|f\| = \|F\| + \delta(\partial D).$$

*Proof.* From the work of §2.5 we know that the least harmonic majorant of  $\log^+ |f|$  is  $u^+ = P[\mu^+]$ , where  $d\mu^+ = \log^+ |F^*| dm + d\delta$ . Let  $v$  be the least harmonic majorant of  $\log(1 + |f|)$ , so that  $v = P[\eta]$  where  $\eta$  is the weak star limit of the measures  $\log(1 + |f_r|) dm$  as  $r \rightarrow 1^-$ . Then the absolutely continuous part of  $\eta$  is

$$\log(1 + |f^*|) dm = \log(1 + |F^*|) dm,$$

so that  $d\eta = \log(1 + |F^*|) dm + d\sigma$ , where  $\sigma$  is a positive measure on  $\partial D$  that is singular with respect to  $m$ . Thus

$$\|f\| = \lim_{r \rightarrow 1^-} \int_{\partial D} \log(1 + |f_r|) dm = \int_{\partial D} d\eta = \|F\| + \sigma(\partial D),$$

so we will be finished if we can prove that  $\sigma = \delta$ .

We take the argument from [17, Proof of Theorem 3.1]. The numerical inequality (1) implies that  $u^+ \leq v \leq u^+ + \log 2$  at each point of  $D$ . These are inequalities on Poisson integrals, and by the uniqueness of Poisson integrals they imply the corresponding inequalities on measures:  $d\mu^+ \leq d\eta \leq d\mu^+ + (\log 2) dm$ . Now inequalities between measures carry over to their singular parts, and since the first and last measures above both have singular part  $\delta$ , the same holds for  $\eta$ .  $\square$

**3.3. Scalar multiplication on  $N$ .** Suppose  $f = F/S_\delta$  as in the proof above. If  $a$  is a complex number, then the corresponding factorization of  $af$  is  $(aF)/S_\delta$ , so by Lemma 3.2  $\|af\| = \|aF\| + \delta(\partial D)$ . But on  $N^+$  scalar multiplication is continuous, hence  $\lim_{a \rightarrow 0} \|aF\| = 0$ . Thus  $\lim_{a \rightarrow 0} \|af\| = \delta(\partial D)$ , which shows that scalar multiplication on  $N$  is discontinuous at the origin. For more on the strange topology of  $N$ , see [17].

The following two corollaries of Lemma 3.2 are essential to the proof of our Main Theorem. For the first, recall that the definition of “bounded set” given in §2.1 is stronger than that of “metrically bounded set.”

**3.4. Proposition.** *Every bounded subset of  $N^+$  is relatively  $\kappa$ -compact.*

*Proof.* Suppose  $A \subset N^+$  is bounded. Fix a sequence  $(f_n)$  in  $A$ . Our goal is to find a subsequence  $(f_{n_k})$  and an  $f \in N^+$  such that  $f_{n_k} \xrightarrow{\kappa} f$ .

By the definition of boundedness, for each  $\epsilon > 0$  there exists  $t_\epsilon > 0$  such that  $A \subset t_\epsilon B_\epsilon^+$ . In particular, the entire sequence  $(f_n)$  lies in the  $\kappa$ -compact (by Lemma 3.1) subset  $t_1 B_1$  of  $N$ . Thus  $f_{n_k} \xrightarrow{\kappa} f$  for some subsequence  $(f_{n_k})$  and some  $f \in N$ . We will be done if we can show that  $f \in N^+$ .

Since  $f \in N$ , §2.5 provides the factorization  $f = F/S_\delta$ , where  $F \in N^+$  and  $S_\delta$  is the singular inner function associated with the singular denominator measure  $\delta$  of  $f$ . We will be done if we can show that  $\delta = 0$  (so that  $S_\delta \equiv 1$ ).

For this, fix  $\epsilon > 0$  and observe that  $t_\epsilon^{-1} f_{n_k} \xrightarrow{\kappa} t_\epsilon^{-1} f$ , and that each  $t_\epsilon^{-1} f_{n_k}$  belongs to  $B_\epsilon^+$ , hence also to  $B_\epsilon$ . Thus the  $\kappa$ -compactness of  $B_\epsilon$  insures that  $t_\epsilon^{-1} f \in B_\epsilon$ . As in §3.3, the denominator measure associated with  $t_\epsilon^{-1} f$  is the same as the one associated with  $f$ , namely  $\delta$ , so Lemma 3.2 shows that  $\delta(\partial D) \leq \|t_\epsilon^{-1} f\| \leq \epsilon$ . Because  $\epsilon$  is an arbitrary positive number, this implies  $\delta(\partial D) = 0$ , as desired.  $\square$

**3.5. Lemma.** *If  $f \in N$  and  $\epsilon > 0$ , then  $f = f_0 f_1 \cdots f_n$ , where  $f_0 \in N^+$  and  $f_1, \dots, f_n \in N$  with  $\|f_j\| < \epsilon$  for  $1 \leq j \leq n$ .*

*Proof.* Choose  $0 < t < 1$  such that  $\log(1 + t) < \epsilon/2$ . Consider once again the Third Fundamental Factorization  $f = F/S_\delta$  provided by §2.5. Choose a positive integer  $n$  so that  $\delta(\partial D)/n \leq \epsilon/2$ , let  $f_0 = t^{-n} F$ , and set  $f_1, \dots, f_n$  all equal to  $t(S_\delta)^{-1/n} = t/(S_{\delta/n})$ . Then  $f$  has the desired product representation, with Lemma 3.2 providing the estimate  $\|f_j\| = \|t\| + \delta(\partial D)/n \leq \epsilon/2 + \epsilon/2 = \epsilon$ .  $\square$

We close this section with an important result about inversion in the Smirnov class that will be needed in the last step of our proof of the Main Theorem.

**3.6. Proposition.** *The mapping  $f \rightarrow 1/f$  is continuous on the set of invertible elements of the Smirnov class.*

*Proof.* Suppose  $\{f_n\}$  is a sequence of invertible elements of  $N^+$ ,  $f \in N^+$  is invertible, and  $f_n \rightarrow f$  in  $N^+$ . Our task is to show that  $1/f_n \rightarrow 1/f$  in  $N^+$ . Now  $f_n^* \rightarrow f^*$  in measure on  $\partial D$ , so  $(1/f_n)^* = 1/f_n^* \rightarrow 1/f^*$  in measure, hence by Lemma 2.4 we need only show that  $\{\log^+(1/|f_n^*|)\}$  is uniformly integrable on  $\partial D$ .

Since all the Smirnov functions involved in this argument are invertible, they are—as we observed in §2.5—outer functions. Hence by (4) we have, for each  $n$ ,

$$\log |f_n(0)| = \int_{\partial D} \log |f_n^*| dm = \int_{\partial D} \log^+ |f_n^*| dm - \int_{\partial D} \log^+(1/|f_n^*|) dm,$$

whereupon

$$\int_{\partial D} \log^+(1/|f_n^*|) dm = \int_{\partial D} \log^+ |f_n^*| dm - \log |f_n(0)|.$$

Now the right-hand side of the last equation converges to

$$\int_{\partial D} \log^+ |f^*| dm - \log |f(0)|$$

by the convergence criterion of Lemma 2.4 and the fact that  $f_n(0) \rightarrow f(0) \neq 0$ . Thus the sequence of non-negative functions  $\{\log^+(1/|f_n^*|)\}$  is uniformly integrable on  $\partial D$ , as we wished to show.  $\square$

#### 4. PROOF OF THE MAIN THEOREM

We have now assembled all the ingredients needed for the proof of Theorem 1.1. We prove only the nontrivial implications, which are (a)  $\rightarrow$  (b)  $\rightarrow$  (c)  $\rightarrow$  (d).

*Proof that (a) $\rightarrow$ (b).* Hypothesis (a) is that  $C_\varphi(B_R^+)$  is bounded for some  $R > 0$ , and we are supposed to prove that  $C_\varphi(N) \subset N^+$ .

The first step is to prove that  $C_\varphi(B_R) \subset N^+$  (remember that  $B_R$  is the closed ball of radius  $R$  in the full Nevanlinna class  $N$ ). Given  $f \in B_R$  consider once more the family of dilates  $\{f_r : 0 \leq r < 1\}$  defined in §2.6. Then  $f_r \in B_R^+$  for each  $r$ , and therefore, by our hypothesis, the image-family  $\{f_r \circ \varphi : 0 \leq r < 1\}$  is a bounded subset of  $N^+$ . Since  $f_r \xrightarrow{\kappa} f$ , we also have  $f_r \circ \varphi \xrightarrow{\kappa} f \circ \varphi$ , so by Proposition 3.4,  $f \circ \varphi \in B_R^+ \subset N^+$ .

So far we know that  $C_\varphi(B_R) \subset N^+$  for a particular  $R > 0$ . Our goal is to show that  $C_\varphi(N) \subset N^+$ . If we could scalar-multiply every function in  $N$  into  $B_R$ , there would be no problem. Unfortunately scalar multiplication is *not* continuous on  $N$  (recall §3.3), however Lemma 3.5 provides a substitute.

Suppose  $f \in N$ . By Lemma 3.5 we can factor  $f$  as  $f_0 f_1 \cdots f_n$ , where  $f_0 \in N^+$  and each  $f_j$  is in  $B_R$ . Thus  $f \circ \varphi = (f_0 \circ \varphi)(f_1 \circ \varphi) \cdots (f_n \circ \varphi)$ . By the first paragraph of this proof  $f_j \circ \varphi \in N^+$  for  $1 \leq j \leq n$ . Since  $f_0 \in N^+$ , we also have  $f_0 \circ \varphi \in N^+$ . Thus  $f \circ \varphi$  is a product of functions in  $N^+$ , and since  $N^+$  is an algebra,  $f \circ \varphi \in N^+$ , as desired.

*Proof that (b) $\leftrightarrow$ (c).* Suppose first that  $C_\varphi(N) \subset N^+$ . We want to show that  $C_\varphi$  is compact on (say)  $H^2$ . For this it is enough, by the previously mentioned result of Sarason (see §2.7), to show that  $C_\varphi(h^1) \subset h_0^1$ . Suppose  $u \in h^1$  with  $u \geq 0$  on  $D$ , and let  $\tilde{u}$  denote the harmonic conjugate of  $u$  that vanishes at the origin. Then  $F = e^{u+i\tilde{u}} \in N$ , so by hypothesis (b),  $F \circ \varphi \in N^+$ . Thus  $u \circ \varphi = \log |F \circ \varphi| \in h_0^1$ ,

as desired. The containment  $C_\varphi(h^1) \subset h_0^1$  follows from this and the fact that every  $u \in h^1$  is the difference of two non-negative members of  $h^1$ .

Conversely, suppose  $C_\varphi$  is compact on  $H^2$ , so by the result of Shapiro and Sundberg, it is compact on  $h^1$ , and therefore by Sarason's theorem,  $C_\varphi(h^1) \subset h_0^1$ . We want to show that  $C_\varphi(N) \subset N^+$ . For this, fix  $f \in N$  and consider its first fundamental factorization  $f = BG$  (§2.5), where  $B$  is the Blaschke product formed with the zeros of  $f$  and  $G \in N$  never vanishes. Then  $f \circ \varphi = (B \circ \varphi)(G \circ \varphi)$ . Since  $B$  is bounded, so is  $B \circ \varphi$ , so we need only show that  $G \circ \varphi \in N^+$ . Now the harmonic function  $u = \log |G|$  belongs to  $h^1$ , so  $u \circ \varphi$  belongs to  $h_0^1$ , and therefore so does its positive part  $\log^+ |G \circ \varphi|$ . This shows that  $G \circ \varphi \in N^+$ , as desired.

*Proof that (b)  $\rightarrow$  (d).* We are given that  $C_\varphi$  is compact on some Hardy space, and we want to show that it is metrically compact on  $N$ , i.e. that it takes any metric ball  $B_R$  into a relatively compact subset of  $N$ . For this, fix  $R > 0$  and suppose  $\{f_n\}$  is a sequence of functions in  $B_R$ . We wish to produce a subsequence  $\{f_{n_k}\}$  such that  $\{f_{n_k} \circ \varphi\}$  converges in  $N$ .

By the second fundamental factorization of §2.5 we may write  $f_n = g_n/h_n$  where  $g_n$  and  $h_n$  are holomorphic in  $D$ , both have modulus bounded by 1 in  $D$ , and  $h_n$  never vanishes on  $D$ . By a normal families argument we may, upon passing to an appropriate subsequence, assume that there exist functions  $g$  and  $h$  holomorphic on  $D$  such that  $g_n \xrightarrow{\kappa} g$  and  $h_n \xrightarrow{\kappa} h$ . Since each  $h_n$  vanishes nowhere on  $D$ , the argument principle demands that  $h$  either vanish everywhere or nowhere. We need to know that it vanishes nowhere.

To prove this, recall that while establishing the second fundamental factorization for Nevanlinna class functions in §2.5 we showed that  $-\log |h_n| = P[\mu_n^+]$  where, for each  $n$ ,  $\mu_n^+$  is a weak star limit of measures  $\log^+ |(G_n)_r| dm$  as  $r \rightarrow 1^-$ , and  $f_n = B_n G_n$  is the first fundamental factorization of  $f_n$ . Thus

$$\mu_n^+(\partial D) = \lim_{r \rightarrow 1^-} \int_{\partial D} \log^+ |(G_n)_r| dm \leq \lim_{r \rightarrow 1^-} \int_{\partial D} \log(1 + |(G_n)_r|) dm,$$

and the limit on the right is just  $\|G_n\| = \|f_n\| \leq R$ , for all  $n$ . Thus all the measures  $\{\mu_n^+\}$  have total variation bounded by  $R$ , which guarantees a subsequence  $\{\mu_{n_k}^+\}$  that converges weak star to a finite positive Borel measure  $\eta$ . Consequently  $-\log |h_{n_k}| = P[\mu_{n_k}^+] \rightarrow P[\eta]$  pointwise on  $D$ . We conclude that  $|h| = \exp\{-P[\eta]\}$  is never zero, as desired.

To this point we know that

$$f_n = g_n/h_n \xrightarrow{\kappa} g/h \stackrel{\text{def}}{=} f,$$

where  $f \in N$ . We finish the proof by showing that  $\|f_n \circ \varphi - f \circ \varphi\| \rightarrow 0$ . Since  $C_\varphi$  is compact on a Hardy space, we know that the radial limit function  $\varphi^*$  has modulus  $< 1$  at almost every point of  $\partial D$  (see [16, §2.2, page 23]), so that  $(g_n \circ \varphi)^* = g_n \circ \varphi^* \rightarrow g \circ \varphi^*$  a.e. on  $\partial D$ . Consequently the Bounded Convergence Theorem insures that  $g_n \circ \varphi \rightarrow g \circ \varphi$  in the metric of  $N^+$ . In exactly the same way,  $h_n \circ \varphi \rightarrow h \circ \varphi$  in  $N^+$ .

For the endgame, note that because neither  $h$  nor any of the  $h_n$  ever vanishes on  $D$ , their reciprocals all belong to  $N$ . Condition (b) insures that the composition of each of these reciprocals with  $\varphi$  belongs to  $N^+$ , so that each  $h_n \circ \varphi$ , as well as  $h \circ \varphi$ , is invertible in  $N^+$ . The continuity of inversion in  $N^+$  (Proposition 3.6), along with the fact that  $h_n \circ \varphi \rightarrow h \circ \varphi$ , now guarantees that  $1/(h_n \circ \varphi) \rightarrow 1/(h \circ \varphi)$

in  $N^+$ . Because  $N^+$  is a topological algebra we therefore have

$$f_n \circ \varphi = (g_n \circ \varphi) \cdot \frac{1}{h_n \circ \varphi} \longrightarrow (g \circ \varphi) \cdot \frac{1}{h \circ \varphi} = f \circ \varphi,$$

where the convergence is in the metric of  $N^+$ . This finishes the proof that (b)  $\rightarrow$  (d), and with it, that of the Main Theorem.  $\square$

*Final remark.* The “linear topological” notion of compactness (wherein the image of *some* ball is relatively compact) can be imported into  $N$ , whereupon it joins the list of equivalences that make up our Main Theorem. Indeed, metric compactness on  $N$  trivially implies compactness, and conversely compactness on  $N$  clearly implies compactness on  $N^+$ , which by our Main Theorem implies (for composition operators) metric compactness on  $N$ .

#### ACKNOWLEDGMENTS

We thank Alec Matheson whose detailed criticisms and insightful suggestions greatly improved this paper. In particular, Matheson suggested the line of argument given here for the implication (b)  $\rightarrow$  (d) of the Main Theorem; our original proof was considerably more complicated. We also wish to thank Wayne Smith for alerting us to several errors in the preprint version of this paper.

#### REFERENCES

- [1] J. S. Choa and H. O. Kim, *Compact composition operators on the Nevanlinna class*, Proc. Amer. Math. Soc. 125 (1997), 145–151. MR **97c**:47031
- [2] J. A. Cima and A. Matheson, *Essential norms of composition operators and Aleksandrov measures*, Pacific J. Math. 179 (1997), 59–64. MR **98e**:47047
- [3] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press 1995. MR **97i**:47056
- [4] P. L. Duren, *Theory of  $H^p$  Spaces*, Academic Press 1970. MR **42**:3552
- [5] T. W. Gamelin, *Uniform Algebras*, Prentice–Hall 1969. MR **53**:14137
- [6] J. B. Garnett, *Bounded Analytic Functions*, Academic Press 1981. MR **83g**:30037
- [7] F. Jafari et al., editors, *Studies on Composition Operators*, Contemp. Math. Vol. 213, American Math. Soc. 1998. MR **98h**:47005
- [8] J. L. Kelley, I. Namioka, et al., *Linear Topological Spaces*, Van Nostrand 1963. MR **29**:3851
- [9] M. I. Masri, *Compact Composition Operators on the Nevanlinna and Smirnov Classes*, Thesis, Univ. of North Carolina at Chapel Hill 1985.
- [10] Y. Nakamura and N. Yanagihara, *Composition operators on  $N^+$* , TRU Math. 14 (1978), 9–16. MR **80f**:30024
- [11] W. Rudin, *Real and Complex Analysis, Third ed.*, McGraw-Hill 1987. MR **88k**:00002
- [12] D. Sarason. *Composition operators as integral operators*, in *Analysis and Partial Differential Equations*, C. Sadowski, ed., Marcel Dekker 1990. MR **92a**:47040
- [13] H. J. Schwartz, *Composition operators on  $H^p$* , Thesis, Univ. of Toledo 1969.
- [14] Jonathan E. Shapiro, *Aleksandrov measures used in essential norm inequalities for composition operators*, J. Operator Th., to appear. CMP 98:17
- [15] J. H. Shapiro, *The essential norm of a composition operator*, Annals of Math. 125 (1987), 375–404. MR **88c**:47058
- [16] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag 1993. MR **94k**:47049
- [17] J. H. Shapiro and A. L. Shields, *Unusual topological properties of the Nevanlinna Class*, American J. Math. 97 (1976) 915–936. MR **52**:11053
- [18] J. H. Shapiro and C. Sundberg, *Compact composition operators on  $L^1$* , Proc. Amer. Math. Soc. 108 (1990), 443–449. MR **90d**:47035
- [19] J. H. Shapiro and P. D. Taylor, *Compact, nuclear, and Hilbert-Schmidt composition operators on  $H^p$* , Indiana Univ. Math. J. 23 (1973), 471–496. MR **48**:4816

- [20] V. I. Smirnov, *Sur les valeurs limites des fonctions, régulière à l'intérieur d'un cercle*, J. Soc. Phys. Math. Léningrad 2, No. 2 (1929), 22–37.
- [21] J. H. Williamson, *Compact linear operators in linear topological spaces*, J. London Math. Soc. 29 (1954) 149–156. MR **15**:801c
- [22] N. Yanagihara, *Multipliers and linear functionals for the class  $N^+$* , Trans. Amer. Math. Soc. 180 (1973), 449–461. MR **49**:3147

DEPARTMENT OF MATHEMATICS EDUCATION, SUNG KYUN KWAN UNIVERSITY, JONGRO-GU, SEOUL 110-745, KOREA

*E-mail address*: `jschoa@yurim.skku.ac.kr`

DEPARTMENT OF MATHEMATICS, KAIST, TAEJON 305-701, KOREA

*E-mail address*: `hkim@ftn.kaist.ac.kr`

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824-1027

*E-mail address*: `shapiro@math.msu.edu`