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ON THE ENTROPY OF THE CONVEX HULL OF FINITE SETS

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ABSTRACT. We give estimates for the entropy numbers and the Gel′fand diameters of the symmetric convex hull of a finite number of points in a Banach or a Hilbert space.

0. INTRODUCTION

Let $(X, \|\cdot\|)$ be a Banach space and let A be a bounded subset of X. The covering numbers $N(A;\varepsilon)$, $\varepsilon > 0$, of A are defined by

$$N(A;\varepsilon) := \inf\left\{ N : \exists\, x_1, \dots, x_N \in M \text{ such that } A \subset \bigcup_{k=1}^{N} B(x_k;\varepsilon) \right\},$$

the entropy numbers $\varepsilon_n(A)$ by

$$\varepsilon_n(A) := \inf\{\varepsilon > 0 : N(A;\varepsilon) \le n\}$$

and the dyadic entropy numbers $e_n(A)$ by

$$e_n(A) := \varepsilon_{2^{n-1}}(A), \qquad n = 1, 2, \dots .$$

The dyadic entropy numbers of a bounded linear operator $u\colon X \to Y$ between two Banach spaces X and Y are defined by

$$e_n(u) := e_n(u(B_X)), \qquad n \in \mathbb{N},$$

where B_X is the unit ball of X.

We say that a Banach space X is of type p, $1 < p \le 2$, if there exists a constant $C > 0$, such that for every finite family of points $\{x_1, x_2, \dots, x_n\}$ of X, we have

$$\int_0^1 \left\| \sum_{i=1}^{n} r_i(t) x_i \right\| dt \le C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p},$$

where $(r_i)_{i=1}^{\infty}$ are the Rademacher functions. The smallest constant C which satisfies the previous inequality is the type p constant of X, which we denote $\tau_p(X)$.

The following theorem is due to B. Maurey (see [P2] and [C]).

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Theorem 0.1. *Let X be a Banach space of type p, $1 < p \leq 2$, and let us consider an operator $S\colon \ell_1^N \to X$. Then there exists a universal constant $C \geq 1$, i.e. C is independent of N, S and X, such that*

$$e_k(S) \leq C\tau_p(X)\left(\frac{\log\left(\frac{N}{k}+1\right)}{k}\right)^{1-\frac{1}{p}}\|S\|, \qquad 1 \leq k \leq N. \tag{1}$$

The Gel′fand numbers $c_n(u)$ of a bounded operator $u\colon X \to Y$ between the Banach spaces X and Y are defined by

$$c_n(u) := \inf\left\{\left\|u\big|_E\right\| : E \subset X,\ \operatorname{codim}(E) < n\right\}.$$

Let H be a Hilbert space and let us consider an operator $S\colon \ell_1^N \to H$. The following estimate is the analogue of (1) for the Gel′fand numbers $c_k(S)$ and it is due to Carl and Pajor (see [CP]):

$$c_k(S) \leq C\sqrt{\frac{\log\left(\frac{N}{k}+1\right)}{k}}\|S\|, \qquad 1 \leq k \leq N. \tag{2}$$

If S is a (closed) subspace $S \subseteq X$, then we denote by Q_S the quotient mapping from X onto X/S. The Kolmogorov numbers $d_n(u)$ of a bounded operator $u\colon X \to Y$ between the Banach spaces X and Y are defined by

$$d_n(u) := \inf\{\|Q_S u\|\colon S \subset X,\ \dim(S) < n\}.$$

We have $c_k(u) = d_k(u^*)$.

Given a set A, we can write down analogues of (1) and (2) which describe the convex hull of A. More precisely, given a set A, we can consider the Banach space $\ell_1(A)$ of all summable families of real numbers, $(\xi_t)_{t\in A}$, indexed over A, i.e.

$$l_1(A) = \left\{(\xi_t)_{t\in A}\colon \|(\xi_t)_{t\in A}\| = \sum_{t\in A}|\xi_t| < +\infty\right\}.$$

We denote by $(e_t)_{t\in A}$ the canonical basis of $\ell_1(A)$.

Now let us assume that A is a bounded subset of a Banach space X and consider the operator $u\colon \ell_1(A) \to X$ defined by $u(e_t) = t$. If $B_{\ell_1(A)}$ is the unit ball of $\ell_1(A)$ and $\operatorname{co}(A)$ is the symmetric convex hull of A, then $u(B_{\ell_1(A)}) = \operatorname{co}(A)$ and hence

$$e_n(\operatorname{co}(A)) = e_n(u) \quad \text{and} \quad c_n(\operatorname{co}(A)) = c_n(u) = d_n(u^*).$$

Suppose now that A is a subset of a Hilbert space H and that $\operatorname{card}(A) = N$. Then the estimates (1) and (2) imply that

$$e_k(\operatorname{co}(A)) \leq C\sqrt{\frac{\log\left(\frac{N}{k}+1\right)}{k}}\operatorname{diam}(A), \qquad 1 \leq k \leq N, \tag{3}$$

$$c_k(\operatorname{co}(A)) \leq C\sqrt{\frac{\log\left(\frac{N}{k}+1\right)}{k}}\operatorname{diam}(A), \qquad 1 \leq k \leq N, \tag{4}$$

where $\operatorname{diam}(A)$ denotes the diameter of A.

If A is a subset of a Banach space X of type p, $1 < p \leq 2$, with $\operatorname{card}(A) = N$, then Theorem 0.1 implies that

$$e_k(\operatorname{co}(A)) \leq C\tau_p(X)\left(\frac{\log\left(\frac{N}{k}+1\right)}{k}\right)^{1-\frac{1}{p}}\operatorname{diam}(A), \qquad 1 \leq k \leq N. \tag{5}$$

We want to obtain variants of the estimates (3), (4) or (5) which take into consideration not only the diameter of A but also the distribution of the points of A in the space, i.e. the decay of the entropy numbers $\varepsilon_n(A)$ of A. This is, for example, the case when $A = \mathrm{co}(\{x_k\}_{k\geq 1})$ where $\{x_k\}_{k\geq 1} \subseteq X$ is a sequence of points in X such that $\|x_k\| \leq \phi(k)$, $k \geq 1$, where ϕ is a positive decreasing function.

In this paper we obtain some results in the case where the entropy numbers $\varepsilon_n(A)$ of A do not decrease too fast as $n \to +\infty$. More precisely, let ϕ be a positive decreasing function on $\mathbb{N}^*$ which satisfies for some $\alpha > 0$,

$$\phi(k) \leq \alpha\phi(2k), \qquad k \geq 1. \tag{6}$$

Let $A \subseteq X$ be a finite subset of a Banach space X with $\mathrm{card}(A) = N$ and let us assume that

$$\varepsilon_k(A) \leq \phi(k), \qquad 1 \leq k \leq N.$$

In this article, we give some estimates of the dyadic entropy numbers $e_k(\mathrm{co}(A))$ and of the Gel′fand numbers $c_k(\mathrm{co}(A))$ in terms of $\phi(k)$. The proofs are inspired by ideas from [BP].

In the first section we treat the case of an arbitrary Banach space X and in the second, the case of a Hilbert space and of a Banach space of type p, $1 < p \leq 2$.

An interesting example of a function ϕ which satisfies (6) is given by

$$\phi(k) = Ck^{-\gamma_1}(\ln(k+1))^{-\gamma_2}, \qquad k = 1, 2, \ldots,$$

for $\gamma_1, \gamma_2 \geq 0$ or for $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}$.

Throughout this article, ϕ will be a positive decreasing function on $\mathbb{N}^*$, satisfying the property (6) for a fixed $\alpha > 0$. Note that (6) easily implies that

$$\phi(m) \leq \alpha^{\left[\log_2 \frac{k}{m}\right]+1}\phi(k), \tag{7}$$

for any $m \leq k$.

1. Case of an arbitrary space

Theorem 1.1. *Let A be a finite subset of the unit ball of a Banach space X with* $\mathrm{card}(A) = N$ *and such that*

$$\varepsilon_k(A) \leq \phi(k), \qquad k = 1, 2, \ldots, N.$$

Then there exists $C(\alpha) > 0$ independent of N, such that

$$c_k(\mathrm{co}(A)) \leq C(\alpha)\phi(k), \qquad 1 \leq k \leq N.$$

Proof. Since A is contained in the unit ball of X, we have

$$c_k(u) \leq c_1(u) \leq 1.$$

So, if we set $C_1 = 1/\phi(6)$, the decay of the function ϕ implies that

$$c_k(\mathrm{co}(A)) \leq 1 \leq C_1\phi(6) \leq C_1\phi(k), \tag{8}$$

for all $k \leq \min\{6, N\}$.

If $6 < k \leq N$, then we set

$$m = \left[\frac{k}{3}\right] \quad \text{and} \quad r = \left[\frac{k}{2}\right].$$

Note that $m \leq k/2$. Let $\Gamma_1 \subseteq A$ be a $2\varepsilon_m(A)$-net of A such that $\mathrm{card}(\Gamma_1) \leq m$. For every $t \in A$, we denote by $\zeta(t)$ an element of Γ_1 such that $\|\zeta(t) - t\| \leq 2\varepsilon_m(A)$ and we define

$$\Gamma_2 = \{t - \zeta(t)\colon t \in A\}.$$

Then $\mathrm{co}(A) \subseteq \mathrm{co}(\Gamma_1) + \mathrm{co}(\Gamma_2)$ and hence, using a classical property of Gel′fand numbers,

$$c_k(\mathrm{co}(A)) \leq c_{r+1}(\mathrm{co}(\Gamma_1)) + c_r(\mathrm{co}(\Gamma_2)). \tag{9}$$

Since $r + 1 > m$ and $\mathrm{card}(\Gamma_1) \leq m$, we have

$$c_{r+1}(\mathrm{co}(\Gamma_1)) = 0. \tag{10}$$

To estimate the remaining term $c_r(\mathrm{co}(\Gamma_2))$, we use the inequality (7) to get

$$c_r(\mathrm{co}(\Gamma_2)) \leq \sup_{s \in \Gamma_2} \|s\| \leq 2\varepsilon_m(A) \leq 2\phi(m) \leq 2\alpha^{[\log_2 \frac{k}{m}]+1}\phi(k).$$

Since $k > 6$, we can see that $\frac{k}{m} \leq 6$, and hence

$$c_r(\mathrm{co}(\Gamma_2)) \leq 2\alpha^{[\log_2 6]+1}\phi(k). \tag{11}$$

Theorem 1.1 follows from (9), (10) and (11) by taking

$$C(\alpha) = \max\{C_1, 2\alpha^{[\log_2 6]+1}\}. \qquad \square$$

The following result is a variant of Carl's inequality (for a proof see [CKP, Theorem 1.3]).

Proposition 1.2. *Let (b_n) be a positive and nondecreasing sequence with the property that there is a constant $\gamma \geq 1$ such that $b_{2n} \leq \gamma b_n$ for all $n \in \mathbb{N}$. Then there exists a constant $c(\gamma) \geq 1$ such that for every bounded $u\colon X \to Y$ between the Banach spaces X and Y and all $n \in \mathbb{N}$,*

$$\sup_{1 \leq k \leq n} b_k e_k(u) \leq c(\gamma) \sup_{1 \leq k \leq n} b_k s_k(u),$$

where s_k denotes either c_k or d_k.

Using Proposition 1.2 and Theorem 1.1, we can obtain the following estimate on the entropy numbers of the symmetric convex hull of A and its dual version.

Theorem 1.3. *Let A be a finite subset of the unit ball of a Banach space X with $\mathrm{card}(A) = N$ and such that*

$$\varepsilon_k(A) \leq \phi(k), \qquad k = 1, 2, \ldots, N.$$

(i) *There exists $C(\alpha) > 0$ independent of N, such that*

$$e_k(\mathrm{co}(A)) \leq C(\alpha)\phi(k), \qquad 1 \leq k \leq N. \tag{12}$$

(ii) *If $u\colon \ell_1(A) \to X$ is the bounded operator defined by $u(e_t) = t$, $t \in A$, there exists $C(\alpha) > 0$ independent of N, such that*

$$e_k(u^*) \leq C_\alpha \phi(k), \qquad 1 \leq k \leq N. \tag{13}$$

Proof of (i). Theorem 1.1 implies that for every integer $n \in \mathbb{N}$,

$$\sup_{k \leq N} \frac{1}{\phi(k)} c_k(\mathrm{co}(A)) \leq C(\alpha).$$

Now we observe that the sequence $b_k = 1/\phi(k)$ is a positive, nondecreasing sequence satisfying $b_{2k} \leq \gamma b_k$, $k \in \mathbb{N}$, with $\gamma = \alpha$.

So, by Proposition 1.2, there exists a function $C'(\alpha) > 0$ such that

$$\sup_{k \leq N} \frac{1}{\phi(k)} e_k(\mathrm{co}(A)) \leq C'(\alpha) \sup_{k \leq N} \frac{1}{\phi(k)} c_k(\mathrm{co}(A)) \leq C'(\alpha)C(\alpha),$$

and hence for all $1 \leq k \leq N$

$$e_k(\mathrm{co}(A)) \leq C'(\alpha)C(\alpha)\phi(k)$$

and the first assertion follows.

Proof of (ii). Since $c_k(u) = d_k(u^*)$, the proof of (ii) follows exactly the same lines. □

Remark. Since the conclusions of Theorems 1.1 and 1.3 do not depend on the cardinal N of A, we can extend these results to the case of arbitrary subsets of X. Thus, we can obtain an elementary proof of a result in [CKP].

The following result is an immediate application of Theorems 1.1 and 1.3.

Corollary 1.4. *Let A be a subset of the unit ball of a Banach space X such that*

$$\varepsilon_k(A) \leq Ck^{-\gamma_1}(\ln(k+1))^{-\gamma_2}, \qquad k = 1, 2, \ldots,$$

for $\gamma_1, \gamma_2 \geq 0$ or for $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}$. Then, there exists a constant $C' = C(\gamma_1, \gamma_2) > 0$ such that

$$\max\{e_k(\mathrm{co}(A)),\ c_k(\mathrm{co}(A))\} \leq C'k^{-\gamma_1}(\ln(k+1))^{-\gamma_2}, \qquad k \geq 1.$$

2. Case of Hilbert spaces

Theorem 2.1. *Let A be a finite subset of the unit ball of a Hilbert space H with* $\mathrm{card}(A) = N$ *and such that*

$$\varepsilon_k(A) \leq \phi(k), \qquad k = 1, 2, \ldots, N.$$

Then there exists $C(\alpha) > 0$ independent of N, such that

$$c_k(\mathrm{co}(A)) \leq C(\alpha)\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\,\phi(k), \qquad 1 \leq k \leq N. \tag{14}$$

Proof. First we observe that, since A is contained in the unit ball of H, we have

$$c_k(\mathrm{co}(A)) \leq c_1(\mathrm{co}(A)) \leq 1.$$

So, if we set $C_1 = \sqrt{6}/(\phi(6)\sqrt{\ln 2})$, then the decay of the function ϕ gives

$$c_k(\mathrm{co}(A)) \leq 1 \leq C_1\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\,\phi(6) \leq C_1\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\,\phi(k), \tag{15}$$

for all $k \leq \min\{6, N\}$.

If $6 < k \leq N$, then we proceed as in the proof of Theorem 1.1 and we set

$$m = \left[\frac{k}{3}\right] \leq \frac{k}{2}, \qquad r = \left[\frac{k}{2}\right].$$

Let us consider $\Gamma_1 \subseteq A$, a $2\varepsilon_m(A)$-net of A, such that $\operatorname{card}(\Gamma_1) \leq m$. For any $t \in A$, we denote by $\zeta(t)$ an element of Γ_1 such that $\|\zeta(t) - t\| \leq 2\varepsilon_m(A)$ and we define the set

$$\Gamma_2 = \{t - \zeta(t) \colon t \in A\}.$$

Then $\operatorname{co}(A) \subseteq \operatorname{co}(\Gamma_1) + \operatorname{co}(\Gamma_2)$, which implies that

$$c_k(\operatorname{co}(A)) \leq c_{r+1}(\operatorname{co}(\Gamma_1)) + c_r(\operatorname{co}(\Gamma_2)). \tag{16}$$

Since $r + 1 > m$ and $\operatorname{card}(\Gamma_1) \leq m$, we obtain

$$c_{r+1}(\operatorname{co}(\Gamma_1)) = 0. \tag{17}$$

To estimate $c_r(\operatorname{co}(\Gamma_2))$, we use Theorem 0.1 and the property (7) of ϕ, to obtain

$$\begin{aligned} c_r(\operatorname{co}(\Gamma_2)) &\leq C_m \sqrt{\frac{\ln\left(\frac{N}{r}+1\right)}{r}} \sup_{s \in \Gamma_2} \|s\| \leq C_M \sqrt{\frac{\ln\left(\frac{N}{r}+1\right)}{r}} 2\varepsilon_m(A) \\ &\leq C_M \sqrt{\frac{\ln\left(\frac{N}{r}+1\right)}{r}} 2\phi(m) \leq C_M \sqrt{\frac{\ln\left(\frac{N}{r}+1\right)}{r}} 2\alpha^{[\log_2 \frac{k}{m}]+1}\phi(k), \end{aligned}$$

where C_M is the numerical constant (independent of N and of k) of Theorem 0.1. Now since $k > 6$, we have $\left[\frac{k}{2}\right] \geq \frac{k}{3}$ and $\frac{k}{m} \leq 6$ and hence

$$\begin{aligned} c_r(\operatorname{co}(\Gamma_2)) &\leq C_M \sqrt{3} \sqrt{\frac{\ln\left(3\frac{N}{k}+1\right)}{k}} 2\alpha^{[\log_2 6]+1}\phi(k) \\ &\leq C_M 6\alpha^{[\log_2 6]+1} \sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}} \phi(k). \end{aligned} \tag{18}$$

The estimates (16), (17) and (18) imply the assertion of the theorem with the constant

$$C(\alpha) = \max\{C_1, 6C_M\alpha^{\log_2 6]+1}\}.$$

□

The following result gives the corresponding estimate for the entropy numbers of the convex symmetric hull of $A \subseteq H$.

Theorem 2.2. *Let A be a finite subset of the unit ball of a Hilbert space H such that* $\operatorname{card}(A) = N$ *and such that*

$$\varepsilon_k(A) \leq \phi(k), \qquad k = 1, 2, \ldots, N.$$

(i) *There exists $C(\alpha) > 0$ independent of N, such that*

$$e_k(\operatorname{co}(A)) \leq C(\alpha) \sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}} \phi(k), \qquad 1 \leq k \leq N. \tag{19}$$

(ii) *If $u \colon \ell_1(A) \to H$ is the bounded operator defined by $u(e_t) = t$, $t \in A$, there exists $C(\alpha) > 0$ independent of N, such that*

$$e_k(u^*) \leq C(\alpha) \sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}} \phi(k), \qquad 1 \leq k \leq N. \tag{20}$$

When $k > N$ a better estimate than (19) is given by the following result ([Pie], [P1], [CS] or [BP]).

Proposition 2.3. *If* $T\colon X \to Y$ *is an operator of rank* n *and* $k \geq n$, *then*

$$e_k(T) \leq 8 \cdot 2^{-\frac{k}{n}} e_n(T).$$

Proof of Theorem 2.2 (i). We shall prove by induction on the cardinal N of A that there exists $C(\alpha) > 0$ such that the following statement, denoted by $\mathbb{P}(N)$, is true:

$$\forall A \subseteq X \text{ such that } \operatorname{card}(A) = N \text{ and such that}$$
$$\forall k \leq N,\ \varepsilon_k(A) \leq \phi(k),$$
$$\text{then } \forall k \leq N,\ e_k(\operatorname{co}(A)) \leq C(\alpha)\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\phi(k).$$

Fix a constant $\beta \geq 3$, to be determined later, and set $C_\beta = \sqrt{2\beta}/(\phi(2\beta)\sqrt{\ln 2})$. Since A is contained in the unit ball of H and since the function ϕ is decreasing, we have

$$\begin{aligned} e_k(\operatorname{co}(A)) \leq e_1(\operatorname{co}(A)) \leq 1 \leq C_\beta\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\phi(2\beta) \\ \leq C_\beta\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\phi(k), \end{aligned} \tag{21}$$

for all $k \leq 2\beta$. So, for all $N \leq 2\beta$, $\mathbb{P}(N)$ holds for $C(\alpha) \geq C_\beta$.

Now let $N_0 > 2\beta$ and suppose that $\mathbb{P}(N)$ is true for all $N < N_0$. We shall prove that $\mathbb{P}(N_0)$ is also true.

If $k \leq 2\beta$, then by (21),

$$e_k(\operatorname{co}(A)) \leq C_\beta\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\phi(k).$$

If $2\beta < k \leq N_0$, then we proceed again as in the proof of Theorem 1.3 and we set

$$m = \left[\frac{k}{\beta}\right] \leq \frac{k}{2}, \qquad r = \left[\frac{k}{2}\right].$$

Next we consider $\Gamma_1 \subseteq A$, a $2\varepsilon_m$-net of A, such that $\operatorname{card}(\Gamma_1) = m$. For any $t \in A$, we choose an element $\zeta(t)$ of Γ_1 such that $\|\zeta(t) - t\| \leq 2\varepsilon_m$ and we define the set

$$\Gamma_2 = \{t - \zeta(t)\colon t \in A\}.$$

Then $\operatorname{co}(A) \subseteq \operatorname{co}(\Gamma_1) + \operatorname{co}(\Gamma_2)$, which implies

$$e_k(\operatorname{co}(A)) \leq e_{r+1}(\operatorname{co}(\Gamma_1)) + e_r(\operatorname{co}(\Gamma_2)).$$

To estimate $e_r(\operatorname{co}(\Gamma_2))$, we use Theorem 0.1 and the inequality (7), to obtain

$$\begin{aligned} e_r(\operatorname{co}(\Gamma_2)) &\leq C_M\sqrt{\frac{\ln\left(\frac{N}{r}+1\right)}{r}}\sup_{s\in\Gamma_2}\|s\| \leq C_M\sqrt{\frac{\ln\left(\frac{N}{r}+1\right)}{r}}2\varepsilon_m(A) \\ &\leq C_M\sqrt{\frac{\ln\left(\frac{N}{r}+1\right)}{r}}2\phi(m) \leq C_M\sqrt{\frac{\ln\left(\frac{N}{r}+1\right)}{r}}2\alpha^{[\log_2\frac{k}{m}]+1}\phi(k), \end{aligned}$$

where C_M is the numerical constant (independent of N and of k) of Theorem 0.1. Since $k > 2\beta$, we can see that $[\frac{k}{2}] \geq \frac{k}{3}$ and $\frac{k}{m} \leq 2\beta$ and hence

$$e_r(\mathrm{co}(\Gamma_2)) \leq 6C_M\alpha^{[\log_2 2\beta]+1}\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\phi(k). \tag{22}$$

To estimate the remaining term $e_{r+1}(\mathrm{co}(\Gamma_1))$, since $\mathrm{card}(\Gamma_1) \leq m < r+1$ and $\frac{r+1}{m} \geq \frac{\beta}{2}$, Proposition 2.3 implies

$$e_{r+1}(\mathrm{co}(\Gamma_1)) \leq 8 \cdot 2^{-\frac{r+1}{m}} e_m(\mathrm{co}(\Gamma_1)) \leq 8 \cdot 2^{-\frac{\beta}{2}} e_m(\mathrm{co}(\Gamma_1)).$$

Now, $\Gamma_1 \subseteq A$ and so $\varepsilon_k(\Gamma_1) \leq \phi(k)$ for all $k \leq N_0$. Moreover, $\mathrm{card}(\Gamma_1) = m$ and $m < N_0$. So by the induction hypothesis, $\mathbb{P}(m)$ is true for Γ_1, which implies

$$e_m(\mathrm{co}(\Gamma_1)) \leq C(\alpha)\sqrt{\frac{\ln 2}{m}}\phi(m).$$

Using again the inequality (7), we have

$$\begin{aligned} e_{r+1}(\mathrm{co}(\Gamma_1)) &\leq 8 \cdot 2^{-\frac{\beta}{2}}\sqrt{\frac{\ln 2}{m}} e_m(\mathrm{co}(\Gamma_1)) \leq 8\sqrt{2}2^{-\frac{\beta}{2}}C(\alpha)\frac{1}{\sqrt{m}}\phi(m) \\ &\leq 8\sqrt{2}2^{-\frac{\beta}{2}}C(\alpha)\alpha^{[\log_2 \frac{k}{m}]+1}\frac{1}{\sqrt{m}}\phi(k). \end{aligned}$$

Since $k > 2\beta$, we have $\frac{k}{m} \leq 2\beta$ and $\left[\frac{k}{\beta}\right] \geq \frac{k}{2\beta}$ and hence

$$e_{r+1}(\mathrm{co}(\Gamma_1)) \leq 8\sqrt{2}2^{-\frac{\beta}{2}}\sqrt{2\beta}\alpha^{[\log_2 2\beta]+1}C(\alpha)\frac{1}{\sqrt{k}}\phi(k).$$

So, we have

$$\begin{aligned} e_k(\mathrm{co}(A)) &\leq e_{r+1}(\mathrm{co}(\Gamma_1)) + e_r(\mathrm{co}(\Gamma_2)) \\ &\leq 16\sqrt{\beta}2^{-\frac{\beta}{2}}\alpha^{[\log_2 2\beta]+1}C(\alpha)\frac{1}{\sqrt{k}}\phi(k) + 6C_M\alpha^{[\log_2 2\beta]+1}\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\phi(k) \\ &\leq \left(16\sqrt{\beta}2^{-\frac{\beta}{2}}\alpha^{[\log_2 2\beta]+1}C(\alpha) + 6C_M\alpha^{[\log_2 2\beta]+1}\right)\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\phi(k). \end{aligned}$$

We now choose β satisfying

$$\beta > 3 \quad \text{and} \quad 16\sqrt{\beta}2^{-\frac{\beta}{2}}(\alpha^{\log_2 2\beta]+1}) < \frac{1}{2}.$$

We set

$$C(\alpha) = C_\beta + 12C_M\alpha^{[\log_2 2\beta]+1}.$$

Then we obtain

$$C > C_\beta \text{ and } C > 16\sqrt{\beta}2^{-\frac{\beta}{2}}\alpha^{[\log_2 2\beta]+1}C(\alpha) + 6C_M\alpha^{[\log_2 2\beta]+1}$$

and consequently

$$e_k(\mathrm{co}(A)) \leq C(\alpha)\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}\phi(k), \qquad 1 \leq k \leq N_0.$$

This proves $\mathbb{P}(N_0)$, and (19) follows.

Remark. We can see that also in this case, the estimates concerning the entropy numbers can be derived from those concerning the Gel′fand numbers, by using Proposition 1.2. We decided to give the above proof because it is more direct.

Proof of Theorem 2.2 (ii). The proof of (20) is similar to the proof of (13). The only difference is that we use the sequence $b_k = \sqrt{k}/(\sqrt{\log(\frac{N}{k}+1)}\phi(k))$ instead of the sequence $b_k = 1/\phi(k)$. We shall omit the details. □

Corollary 2.4. *Let A be a finite subset of the unit ball of a Hilbert space H with* $\operatorname{card}(A) = N$ *and such that*

$$\varepsilon_k(A) \leq Ck^{-\gamma_1}(\ln(k+1))^{-\gamma_2}, \qquad k = 1, 2, \ldots, N,$$

for $\gamma_1, \gamma_2 \geq 0$ or for $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}$. Then there exists a constant $C' = C(\gamma_1, \gamma_2) > 0$ such that

$$\max\{e_k(\operatorname{co}(A)), c_k(\operatorname{co}(A))\} \leq C'\sqrt{\frac{\ln\left(\frac{N}{k}+1\right)}{k}}k^{-\gamma_1}(\ln(k+1))^{-\gamma_2}, \qquad 1 \leq k \leq N.$$

The following result is the analogous version to Theorem 2.2 for Banach spaces of type p, $1 < p \leq 2$, and can be proved in the same way.

Theorem 2.5. *Let A be a finite subset of the unit ball of a Banach space X of type p, $1 < p \leq 2$, with* $\operatorname{card}(A) = N$ *and such that*

$$\varepsilon_k(A) \leq \phi(k), \qquad k = 1, 2, \ldots, N.$$

(i) *There exists a constant $C = C(p, \alpha) > 0$ such that*

$$e_k(\operatorname{co}(A)) \leq C\left(\frac{\ln\left(\frac{N}{k}+1\right)}{k}\right)^{1-\frac{1}{p}}\phi(k), \qquad 1 \leq k \leq N.$$

(ii) *If $u\colon \ell_1(A) \to X$ is the bounded operator defined by $u(e_t) = t$, $t \in A$, there exists a constant $C = C(p, \alpha) > 0$ such that*

$$e_k(u^*) \leq C\left(\frac{\ln\left(\frac{N}{k}+1\right)}{k}\right)^{1-\frac{1}{p}}\phi(k), \qquad 1 \leq k \leq N.$$

Corollary 2.6. *Let A be a finite subset of the unit ball of a Banach space X of type p, $1 \leq p \leq 2$, with* $\operatorname{card}(A) = N$ *and such that*

$$\varepsilon_k(A) \leq Ck^{-\gamma_1}(\ln(k+1))^{-\gamma_2}, \qquad k = 1, 2, \ldots, N,$$

for $\gamma_1, \gamma_2 \geq 0$ or for $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}$. Then there exists a constant $C' = C(p, \gamma_1, \gamma_2) > 0$ such that

$$e_k(\operatorname{co}(A)) \leq C'\left(\frac{\ln\left(\frac{N}{k}+1\right)}{k}\right)^{1-\frac{1}{p}}k^{-\gamma_1}(\ln(k+1))^{-\gamma_2}, \qquad 1 \leq k \leq N.$$

Remark. We could not prove an analogue of Theorem 2.5 (i) for Gel′fand numbers, because there does not exist an estimate similar to the estimate (2) for Banach spaces of type p, $1 < p \leq 2$. Nevertheless, we can prove such a result for the smaller class of Banach spaces $X = [X_1, H]_{\frac{2p-2}{p},p}$, i.e. the $(\frac{2p-2}{p}, p)$-interpolates of a Banach space X_1 and a Hilbert space H, by using a result on the interpolation of Gel′fand numbers [Pie, Proposition 11.5.8].

Let us recall that for $K \subseteq \mathbb{R}^n$, the polar set K° of K is defined by

$$K^\circ = \{x \in \mathbb{R}^n \colon \langle x, y \rangle \leq 1 \ \forall y \in K\}.$$

The Santaló inequality [S] and the inverse Santaló inequality [BM] compare the volume of a compact symmetric convex set $B \subseteq \mathbb{R}^n$ with a non-empty interior with the volume of its polar B°, in the following way:

There are positive constants D_1 and D_2 (independent of n and B) such that

$$D_1/n \leq (\operatorname{vol}(B)\operatorname{vol}(B^\circ))^{\frac{1}{n}} \leq D_2/n. \tag{23}$$

Using the above inequalities, the previous results concerning the entropy numbers of $\operatorname{co}(A)$ can provide volumic estimates for $\operatorname{co}(A)$ and its polar $(\operatorname{co}(A))^\circ$.

Proposition 2.7. *Let H be an n-dimensional Hilbert space and let A be a finite subset of its unit ball with $\operatorname{card}(A) = N \geq n$ and such that*

$$\varepsilon_k(A) \leq \phi(k), \qquad k = 1, 2, \ldots, N.$$

Then there exists a constant $C = C(\alpha) > 0$, independent of n, N and A, such that

$$(\operatorname{vol}_n(\operatorname{co}(A))^\circ)^{\frac{1}{n}} \geq C \left(\ln\left(\frac{N}{n} + 1\right)\right)^{-\frac{1}{2}} \phi(n)^{-1}. \tag{24}$$

If A is a finite subset of the unit ball of ℓ_p^n, $1 < p \leq 2$, and under the same hypothesis, then there exists $C = C(\alpha, p) > 0$ such that

$$(\operatorname{vol}_n(\operatorname{co}(A))^\circ)^{\frac{1}{n}} \geq C \left(\ln\left(\frac{N}{n} + 1\right)\right)^{-1+\frac{1}{p}} \phi(n)^{-1}. \tag{25}$$

Proof. By Theorem 2.2 we have that for any $k \leq N$,

$$e_k(\operatorname{co}(A)) \leq C_1(\alpha) \sqrt{\frac{\ln\left(\frac{N}{k} + 1\right)}{k}}\, \phi(k).$$

In volumic terms, this means that

$$(\operatorname{vol}_n(\operatorname{co}(A)))^{\frac{1}{n}} \leq 2C_1(\alpha) \sqrt{\frac{\ln\left(\frac{N}{n} + 1\right)}{n}}\, \phi(n) \frac{C_2}{\sqrt{n}}. \tag{26}$$

Using (26) and (23), we obtain that

$$(\operatorname{vol}_n(\operatorname{co}(A))^\circ)^{\frac{1}{n}} \geq \frac{1}{(\operatorname{vol}_n(\operatorname{co}(A)))^{\frac{1}{n}}} \frac{D_1}{n} \geq (2C_1(\alpha)C_2D_1) \Bigg/ \left(\sqrt{\ln\left(\frac{N}{n} + 1\right)}\phi(n)\right)$$

which proves (24). The inequality (25) can be proved in the same way. $\square$

A result similar to that of Proposition 2.7 has been given in [BP2].

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