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ANNIHILATING A SUBSPACE OF L_1 WITH THE SIGN OF A CONTINUOUS FUNCTION

DANIEL WULBERT

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Abstract. Let (X, Σ, μ) be a σ-finite, nonatomic, Baire measure space. Let G be a finite dimensional subspace of $L_1(X, \Sigma, \mu)$. There is a bounded, continuous function, q, defined on X, such that
(1) $\int_X g \operatorname{sgn} q d\mu = 0$ for all $g \in G$, and (2) $|\operatorname{sgn} q| = 1$ almost everywhere.

1. Introduction

Let (X, Σ, μ) be a finite, nonatomic measure space, and let G be a finite dimensional subspace of $L_1(X, \Sigma, \mu)$. The Phelps-Dye theorem states that there is a measurable function, s, such that $|s| = 1$ almost everywhere, and $\int_X sgd\mu = 0$ for all $g \in G$. This is equivalent to saying that there is an extreme point of the unit ball of the dual space of L_1 that annihilates G. The theorem was originally used by Phelps in 1960, to show that no finite-dimensional subspace of $L_1(X, \Sigma, \mu)$ could admit unique best approximations to all integrable functions (also see [11]).

This paper shows that with the existence of a related topology on X, the theorem can be strengthened so that s is also the sign of a continuous function.

There are other recent results involving an ambient topology in a measure space setting. For example, Liapanov's theorem states that the range of a finite, nonatomic, vector-valued measure is a compact convex set. H.Render and H.Stroetmann [12] find necessary conditions that the range be unchanged when the measure is restricted to only open sets. The strengthening of the Liapanov theorem (unlike here) is not always possible.

A significant step in the proof here is the construction of a continuous, bounded function whose level sets have measure zero on a σ-finite, nonatomic, Baire measure space. The main results in this paper are Theorem 5.3, and Theorem 4.3 and its Corollary 4.4.

The paper assumes no regularity conditions on the measure space or on the topology.

2. Notation and definitions

Measure spaces: Throughout the paper (X, Σ, μ) will be a measure space. For $U, K \in \Sigma$, we say that U *splits* K if $\mu(K) > \mu(K \cap U) > 0$. A set $K \in \Sigma$ is an *atom*

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of μ if $\mu(K) > 0$ and no U in Σ splits K. (X, Σ, μ) is *nonatomic* if no set in Σ is an atom with respect to μ. We use *a.e.* as an abbreviation for the term "almost everywhere". We will reserve S for a collection of sets that generate Σ.

Functions: Let f be a real-valued function defined on X. The *support* of f is *supp*$(f) := \{x : f(x) \neq 0\}$. For $A \subseteq \mathbf{R}$, $f^{-1}(A) := \{x \in X : f(x) \in A\}$. The *a-level set* of f is $f^{-1}(a) := f^{-1}(\{a\})$. The 0-level set of f is abbreviated to $Z(f)$. The sign of a function f is $\operatorname{sgn} f(x) := \{1 \text{ if } f(x) > 0; -1 \text{ if } f(x) < 0; \text{ and } 0 \text{ if } f(x) = 0\}$. The composite of two functions f and g is written $f \circ g$, i.e. $f \circ g(x) = f(g(x))$.

Sets: A set in the smallest σ-algebra containing the supports of all the nonnegative continuous real function is a *Baire set*. For sets K and U, we use $K - U$ to represent $\{x \in K : x \notin U\}$, and the *complement* of $K \subseteq X$ is $K^c := X - K$.

3. NONATOMIC SETS

Lemma 3.1. *For a set $K \in \Sigma$, put $M = \{V \in \Sigma : V$ does not split $K\}$. If $\mu(K) < \infty$, then:*

(i) *M is a monotone class,*
(ii) *if $A \in M$, then $A^c \in M$, and*
(iii) *M is a σ-algebra.*

Proof. Most of the properties are trivial to verify. We observe only that since

$$\mu(K \cap [U \cup V]) + \mu(K \cap [U \cap V]) = \mu(K \cap U) + \mu(K \cap V),$$

if $\infty > \mu(K) = \mu(K \cap U) = \mu(K \cap V)$, then $\mu(K \cap [U \cap V]) = \mu(K)$. Therefore since $\infty > \mu(K)$, M is an algebra of sets, and the Monotone Class Theorem implies that M is a σ-algebra. ☐

For the remainder of the paper we will reserve S to be a collection of sets that generates Σ. Although we will often state our results for an abstract set S, the collection of sets to which we later apply the results is the set of supports of continuous functions.

Lemma 3.2. *Let (X, Σ, μ) be nonatomic. If K be a set in Σ such that $\infty > \mu(K) > 0$, then there is a $U \in S$ that splits K.*

Proof. Since μ is nonatomic, some member of Σ splits K. Since S generates Σ, the lemma follows from Lemma 3.1 (iii). ☐

Example. The conclusions of Lemmas 3.1 and Lemma 3.2 are not necessarily true for sets K of infinite measure. Easy examples can be constructed with Lebesgue measure on the real line.

For the following lemma let S be a collection of sets that is closed under finite unions and finite intersections. Assume also that S generates Σ and that μ is nonatomic.

Lemma 3.3. *If $V \in S$ and $\mu(V) < \mu(X) < \infty$, then*

$$\inf\{\mu(U) : U \supseteq V, U \in S, \text{and } \mu(U) > \mu(V)\} = \mu(V).$$

Proof. First we observe that there is a set $U_0 \in S$ such that $V \subseteq U_0$ and $\mu(V) < \mu(U_0) < \infty$. That is if U is a member guaranteed by Lemma 3.2 that splits V^c, then $U_0 = U \cup V$ has this property.

Now having chosen $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_{n-1} \supseteq V$ with each $U_i \in S$, let

$$u_n = \inf\{\mu(U) : U_{n-1} \supseteq U \supseteq V, U \in S \text{ and } \mu(U) > \mu(V)\}.$$

Choose $U_n \in S$ so that $U_{n-1} \supseteq U_n \supseteq V$ and $u_n + \frac{1}{n} > \mu(U_n) \geq u_n$. Note that $u_1 \leq u_2 \leq u_3 \leq \ldots$.

Let $K = \bigcap_{i=1}^{\infty} U_i$. We have that $V \subseteq K$, $K - V \in \Sigma$, and $\lim \mu(U_i) = \mu(K) = \lim u_i \geq \mu(V)$. If $\mu(K) > \mu(V)$, then from Lemma 3.2 there is a $W \in S$ that splits $K - V$.

Choose m large enough that $\frac{1}{m} < \mu(K - V) - \mu((K - V) \cap W)$.

Now

(i) $V \subseteq (V \cup W) \cap U_m \subseteq U_m$,
(ii) $\mu([V \cup W] \cap U_m) \geq \mu([V \cup W] \cap K) = \mu(V) + \mu((K - V) \cap W) > \mu(V)$, and
(iii) $\mu([V \cup W] \cap U_m) \leq \mu(U_m) - [\mu(K - V) - \mu((K - V) \cap W)] < \mu(U_m) - \frac{1}{m}$.

So the set $(V \cup W) \cap U_m \in S$ contradicts the choice of the set U_m. □

In addition to S and (X, Σ, μ) having the properties of the last lemma, for the corollary below, assume also that S is closed under countable unions.

Corollary 3.4. *If $\infty > \mu(X)$, and $c \in [0, \mu(X)]$, there is a $U_c \in S$ such that $\mu(U_c) = c$.*

Proof. Consider all collections of sets $\{U_i\}_{i \in I} \subseteq S$ where: (i) $I \subseteq \mathbf{R}$, and $\sup_{i \in I} i \leq c$, (ii) $i < j$ implies that $U_i \subseteq U_j$, and (iii) $\mu(U_i) = i$. We can order such subsets by inclusion. That is $\{U_i\}_{i \in I} \leq \{V_i\}_{i \in J}$ if and only if $I \subseteq J$ and $U_i = V_i$ for all $i \in I$.

By taking a union we see that every totally ordered set of these collections has a maximal element. By Zorn's lemma the entire set has maximal elements. Let $\{U_i\}_{i \in I}$ be such a maximal element. Let $b = \sup\{i : i \in I\}$.

Let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence of real numbers decreasing to zero such that $b - \delta_n \in I$. Then $\bigcup_{n=1}^{\infty} U_{b-\delta_n}$: (i) is in S, (ii) contains $\bigcup_{j<b, j \in I} U_j$, and (iii) has measure b. So we may assume that $b \in I$. If $b < c$, then Lemma 3.3 provides a set $V \in S$ such that $U_b \subseteq V$ and $b < \mu(V) < c$. This, however, would contradict the maximality of $\{U_i\}_{i \in I}$. □

Comment. This corollary will be superseded by Corollary 4.4. Corollary 4.4 will exhibit a nested collection of open sets with the properties above. However, we use Corollary 3.4 in the development of the argument leading to Corollary 4.4.

4. FUNCTIONS WITH LEVEL SETS OF ZERO MEASURE

Lemma 4.1. *Let $\mu(X) < \infty$. Let f and g be measurable functions. Except for possibly countably many values of r, every level set of $f + rg$ is contained, almost everywhere, in the intersection of a level set of f and a level set of g.*

Proof. There are at most countably many sets of the form $\Lambda_i = f^{-1}(a_i) \cap g^{-1}(b_i)$ that have positive measure. Let $\Lambda = \bigcup_{i=1}^{\infty} \Lambda_i$. Let r and s be distinct, nonzero, real numbers. If a set is contained in the intersection of level sets of two of the functions f, g, $f + rg$, and $f + sg$, then it is also contained in a level set of each of the other two. Hence for all a and b in $\mathbf{R}$,

$$\mu([(f + rg)^{-1}(a) \cap \Lambda^c] \cap [(f + sg)^{-1}(b) \cap \Lambda^c]) = 0.$$

Since $\mu(X) < \infty$, for any fixed $a \in \mathbf{R}$, there can be at most countably many values of r for which

$$\mu[(f + rg)^{-1}(a) \cap \Lambda^c] > 0.$$

Hence, except for these countably many values of r, if $\mu[(f + rg)^{-1}(a)] > 0$, then $(f + rg)^{-1}(a) \subseteq \Lambda$ a.e.

We still need to show that $(f + rg)^{-1}(a)$ is contained a.e. in a single set $\Lambda_i = f^{-1}(a_i) \cap g^{-1}(b_i)$. If $(f + rg)^{-1}(a)$ intersects both Λ_i and Λ_j, then $a_k + rb_k = a$ for $k = i$ and j. Hence $b_j \neq b_i$, and $r = \frac{a_i - a_j}{bj - b_i}$. Hence if r also does not assume any of the possibly countably many values $\{\frac{a_i - a_j}{bj - b_i}\}_{i=1,j=1,i\neq j}^{\infty,\infty}$, then $(f + rg)^{-1}(a)$ is contained a.e. in a single set Λ_i. □

Lemma 4.2. *Let (X, Σ, μ) be a finite, nonatomic Baire measure space. There is a bounded, nonnegative, real-valued, continuous function, f, for which $\mu(f^{-1}(a)) = 0$ for all values of a.*

Comment. Let S represent the collection of supports of bounded, nonnegative, real-valued, continuous functions defined on X. The proof is a construction of the function with the properties of the lemma statement.

Proof. Let f_0 be a nonnegative, continuous function whose maximum is 1. A function can have at most countably many level sets of positive measure. Order by size the level sets $\{\Lambda_{0,i}\}_{i=1}^{\infty}$ of f_0 which have positive measure. That is, $i \leq j$ implies $\mu(\Lambda_{0,i}) \geq \mu(\Lambda_{0,j})$. Put $r_0 = 1$.

Now suppose we have chosen continuous, bounded, nonnegative functions $\{f_0, f_1, \cdots, f_{n-1}\}$; positive numbers $\{r_0, r_1, \cdots, r_{n-1}\}$; and sets $\{V_1, V_2, \cdots, V_n\} \subseteq S$ with the properties listed below.

Let $s_k = \sum_{i=0}^{k} r_i f_i$. Let $\{\Lambda_{k,i}\}_{i=1}^{\infty}$ be the level sets of s_k which have positive measure. As before let these level sets be ordered by size. Let $\Lambda_k = \bigcup_{i=1}^{\infty} \Lambda_{k,i}$. The properties we assume for $\{f_i\}_{i=1}^{n-1}, \{r_i\}_{i=1}^{n-1}$, and $\{V_i\}_{i=1}^{n}$ are that, for any fixed k:

1. for each i there is a j such that $\Lambda_{k,i} \subseteq \Lambda_{k-1,j}$,
2. if $\Lambda_{k,j} \subseteq \Lambda_{k-1,1}$, then $\mu(\Lambda_{k,j}) \leq \frac{1}{2}\mu(\Lambda_{k-1,1})$,
3. $\|r_k f_k\|_\infty \leq \frac{1}{2^k}$,
4. $\Lambda_{k-1} \subseteq V_k \subseteq V_{k-1}$,
5. $\mu(V_k - \Lambda_{k-1}) \leq \frac{1}{k}$, and
6. $\mathit{supp} f_k \subseteq V_k$.

We proceed to choose f_n, r_n and V_{n+1}. From Corollary 3.4 there is a bounded continuous function f_n such that:

$$\mu(\Lambda_{n-1} \cap \mathit{supp} f_n) = \frac{1}{2}\mu(\Lambda_{n-1}).$$

That is, to apply Corollary 3.4 we observe that if χ is the σ-algebra of subsets of Λ_{n-1} generated by $\{U \cap \Lambda_{n-1} : U \in S\}$, then $(\Lambda_{n-1}, \chi, \mu)$ is nonatomic (e.g., use Lemma 3.2). Applying Corollary 3.4 to this measure space yields a generating set $U_{\frac{1}{2}} \cap \Lambda_{n-1}$ (with $U_{\frac{1}{2}} \in S$) such that $\mu(U_{\frac{1}{2}} \cap \Lambda_{n-1}) = \frac{1}{2}\mu(\Lambda_{n-1})$.

Let v be a continuous, nonnegative, bounded function with support V_n. Replacing f_n with vf_n, if necessary, we may assume that $\mathit{supp} f_n \subseteq V_n$.

Now consider $s_n = s_{n-1} + rf_n$. From Lemma 4.1, there are at most countably many real numbers r such that $s_{n-1} + rf_n$ has a level set that intersects Λ_{n-1}^c in a set of positive measure. We will choose r_n to satisfy 4 conditions.

(i) $\mu(s_n^{-1}(a) \cap \Lambda_{n-1}^c) = 0$ for all a.

(ii) If $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ are all the real numbers such that, $\mu(s_{n-1}^{-1}(a_i)) > 0$ and $\mu(f_n^{-1}(b_i)) > 0$, then we require that

$$r_n \notin \{\frac{a_k - a_i}{bj - b_l}\}_{i,j,k,l=1;j \neq l}^{\infty,\infty,\infty,\infty}.$$

Therefore $a_i + r_n b_j \neq a_k + r_n b_l$ and each of the sets $s_{n-1}^{-1}(a_i) \cap f_n^{-1}(b_j)$ are distinct level sets of $s_n = s_{n-1} + r f_n$.

(iii) $\|r_n f_n\|_\infty < \frac{1}{2}\|r_{n-1} f_{n-1}\|_\infty$.

To state the fourth condition, let M_n be chosen so that

$$\sum_{i=M_n}^{\infty} \mu(\Lambda_{n-1,i}) < \frac{1}{n-1},$$

and let

$$\epsilon_n = \inf\{|s_{n-1}(u) - s_{n-1}(v)| : u \in \Lambda_{n-1,i}, v \in \Lambda_{n-1,j}, i \neq j, \text{ and } i, j < M_n\}.$$

We note that $\epsilon_n > 0$.

(iv) $\|r_n f_n\|_\infty < \frac{1}{4}\epsilon_n$.

We now choose V_{n+1}. Let $\{a_i\}$ be the countable set of real numbers so that $s_n^{-1}(a_i) = \Lambda_{n,i}$. For each i there is a d_i such that

$$\mu\left(s_n^{-1}\big((a_i - d_i, a_i) \cup (a_i, a_i + d_i)\big)\right) \leq \frac{1}{n2^i}.$$

Let

$$V_{n+1} = \left[\bigcup_{i=1}^{\infty} s_n^{-1}\big((a_i - d_i, a_i + d_i)\big)\right] \cap V_n.$$

Then V_{n+1} is in S and satisfies conditions (4), (5), and (6).

Finally put $f = \sum_{i=1}^{\infty} r_i f_i$. We have that f is a nonnegative, continuous (condition (3)), real-valued function. The remainder of the proof is to show that $\mu(f^{-1}(a)) = 0$ for all real values a. We will show that, for any $\eta > 0$, $\mu(f^{-1}(a)) < \frac{1}{\eta}$.

Condition (i) implies that $\Lambda_n \subseteq \Lambda_{n-1}$. Condition (ii) and condition (2) guarantee that:

$$\lim_{k\to\infty} \big[\sup\{\mu(\Lambda_{k,i}) : i = 1, 2, \ldots, \infty\}\big] = \lim_{k\to\infty} \mu(\Lambda_{k,1}) = 0.$$

Applying (vi) and (iii) to $n+1$ (instead of n) implies that f takes distinct values on $\Lambda_{n,i}$ and $\Lambda_{n,j}$, for $i \neq j$, and $i, j < M_{n+1}$. By the choice of M_n,

$$\mu(f^{-1}(a) \cap \Lambda_n) \leq \mu(\Lambda_{n,1}) + \mu\left(\bigcup_{i=M_{n+1}}^{\infty} \Lambda_{n,i}\right) \leq \mu(\Lambda_{n,1}) + \frac{1}{n} \to 0.$$

Choose N large enough that $\mu(f^{-1}(a) \cap \Lambda_N) < \frac{1}{3\eta}$ and that $\frac{1}{N} < \frac{1}{3\eta}$. We will estimate the measure of the portions of $f^{-1}(a)$ in the three sets $\Lambda_N, V_{N+1} - \Lambda_N$, and V_{N+1}^c. We already have the first estimate.

The second estimate follows from condition (5),

$$\mu\big(f^{-1}(a) \cap [V_{N+1} - \Lambda_N]\big) \leq \mu(V_{N+1} - \Lambda_N) < \frac{1}{3\eta}.$$

For the third set, we observe from condition (6) that if $x \in V_{N+1}^c$, then $f_j(x) = 0$ for all $j \geq N + 1$. That is, $f(x) = \sum_{i=1}^{N} r_i f_i(x) = s_N(x)$. From the definition of Λ_N, s_N has no level set that intersects Λ_N^c in a set of positive measure. Since $V_{N+1}^c \subseteq \Lambda_N^c$ and that $f = s_N$ on V_{N+1}^c, we conclude that $\mu(f^{-1}(a) \cap V_{N+1}^c) = 0$.

Therefore putting together the three estimates we have,

$$\begin{aligned} 0 &\le \mu(f^{-1}(a)) = \mu(f^{-1}(a) \cap \Lambda_N) + \mu(f^{-1}(a) \cap [V_{N+1} - \Lambda_N]) + \mu(f^{-1}(a \cap V_{N+1}^c)) \\ &< \frac{1}{3\eta} + \frac{1}{3\eta} + 0. \end{aligned}$$

□

Theorem 4.3. *If $(X, \sum, \mu)$ is a σ-finite, nonatomic, Baire measure space, there is a bounded, nonnegative, continuous, real-valued function f whose level sets have zero measure.*

Proof. Suppose that $\{X_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets, each of finite positive measure, and which union to X. We define a new measure,

$$\nu(K) = \sum_{i=1}^{\infty} \frac{\mu(K \cap X_i)}{2^i \mu(X_i)}.$$

Then ν is a finite, nonatomic, Baire measure with the same null sets as μ. Therefore the function of Lemma 4.3 that has level sets of zero ν measure also has level sets of zero μ measure. □

Notation. Let $\operatorname{cl} U$ represents the closure of a set U.

Corollary 4.4. *If (X, Σ, μ) is a finite, nonatomic, Baire measure space, there is a collection of sets $\{U_i\}_{0 \le i \le \mu(X)}$ such that: (1) each U_i is an open set that is the support of a nonnegative, continuous function, (2) $i < j$ implies that $\operatorname{cl} U_i \subseteq U_j$, and (3) $\mu(U_i) = \mu(\operatorname{cl} U_i) = i$.*

Proof. Let f be a continuous, nonnegative, bounded function from Theorem 4.3. The open sets $V_a = f^{-1}\big((a, \infty)\big)$ for $0 \le a \le \|f\|_\infty$ form a collection such that (1) each is the support of a nonnegative, continuous function, (2a) $b > a$ implies that $\operatorname{cl} V_b \subseteq V_a$, (2b) $\mu(V_a) = \mu(\operatorname{cl} V_a)$, and (3) $\mu(V_a)$ assumes every value between 0 and $\mu(X)$. To see that (3) is true, let $0 < \alpha < \mu(X)$. Let $A = \{a : \mu(V_a) \ge \alpha\}$ and let $B = \{a : \mu(V_a) < \alpha\}$. Since every number is either in A or B, the infimum of the numbers in B is equal the supremum of those in A. Call that common number b. Then both

$$\mu[f^{-1}\big([b, \infty)\big)] = \mu[\bigcap_{a \le b} f^{-1}\big((a, \infty)\big)] = \lim_{a \to b, a \in A} \mu[f^{-1}\big((a, \infty)\big)] \ge \alpha,$$

and

$$\mu[f^{-1}\big((b, \infty)\big)] = \mu[\bigcup_{a > b} f^{-1}\big((a, \infty)\big)] = \lim_{a \to b, a \in B} \mu[f^{-1}\big((a, \infty)\big)] \le \alpha.$$

Since $\mu(f^{-1}(b)) = 0$,

$$\alpha \le \mu[f^{-1}\big([b, \infty)\big)] = \mu[f^{-1}\big((b, \infty)\big)] \le \alpha.$$

Let U_i be a set V_a such that $\mu(V_a) = i$. (Note: it is possible that $V_a = V_b$ for distinct a and b.) □

5. The generalized Phelps-Dye theorem

Definition. Let (X, Σ, μ) be a measure space. A linear space Q of measurable functions is called *smooth* if the zero set, $Z(q) = q^{-1}(0)$, of each $q \in Q$ has measure zero.

Comment. This use of the term "smooth" for a subspace $Q \subseteq L_1(X, \Sigma, \mu)$ corresponds to common Banach space usage. That is, Q is smooth in the sense here if and only if for each $q \in Q$ with $\|q\|_1 = 1$, there is a unique bounded linear functional l_q defined on L_1 such that $l_q(q) = 1 = \|l_q\|$, and hence Q is a smooth (in the Banach space sense) subspace of L_1. Of course, $l_q f = \int_X f(\operatorname{sgn} q)d\mu$. However, for our purposes, the smooth subspaces Q are not necessarily contained in L_1.

Lemma 5.1. *Let (X, Σ, μ) be a measure space. Let G be an n-dimensional subspace of $L_1(X, \Sigma, \mu)$. Let Q be an $n + 1$-dimensional, smooth linear space of bounded measurable functions. There is a $q \in Q$ such that, for all $g \in G$,*

$$\int_X g(\operatorname{sgn} q)d\mu = 0.$$

Proof. Let $\{g_1, g_2, \ldots, g_n\}$ be a basis for G. For $q \in Q$ put

$$L(q) = (\int_X g_1 \operatorname{sgn} q d\mu, \int_X g_2 \operatorname{sgn} q d\mu, \ldots, \int_X g_n \operatorname{sgn} q d\mu).$$

We first observe that L is continuous. Suppose that q_i converges uniformly to q_0, and that $g \in G$. Then $|g \operatorname{sgn} q_i| \leq |g| \in L_1$. Also since $\mu[Z(q_0)] = 0$, $q_i \to q_0$ (uniformly) implies that $g \operatorname{sgn} q_i$ converges pointwise a.e. to $g \operatorname{sgn} q_0$. From the dominated convergence theorem, $\int_X g \operatorname{sgn} q_i d\mu$ converges to $\int_X g \operatorname{sgn} q_0 d\mu$ and L is continuous.

Restricting L to $\{q \in Q : \|q\|_\infty\}$ we get a continuous, antipodal (i.e. $L(-q) = -L(q)$) mapping of the surface of an $n+1$-dimensional sphere into an n-dimensional space. From the Borsuk antipodal mapping theorem, there is a $q \in Q$ such that $L(q) = 0$ and therefore satisfies the statement of the lemma. $\square$

Lemma 5.2. *Let (X, Σ, μ) be a σ-finite, nonatomic, Baire measure space. For every integer n there is an n-dimensional smooth space of bounded, continuous functions defined on X.*

Proof. Let Π_k be the polynomials of degree k or less. Let f be the function of Theorem 4.3. Then $Q = \{p \circ f : p \in \Pi_{n-1}\}$ satisfies the conditions of the lemma. $\square$

Theorem 5.3. *Let (X, Σ, μ) be a nonatomic, Baire measure space, and let G be a finite dimensional subspace of $L_1(X, \Sigma, \mu)$. Let U be a σ-finite set in Σ such that suppg $\subseteq U$ for all $g \in G$. (e.g., $G =$ suppG.) There is a bounded, continuous function, q, defined on X such that*

1. $\int_X g \operatorname{sgn} q d\mu = 0$ *for all $g \in G$, and*
2. $|\operatorname{sgn} q| = 1$ *almost everywhere on U.*

Proof. For each Baire set K, put $\nu(K) = \mu(K \cap U)$. Then ν is a nonatomic, σ-finite, Baire measure on X. Furthermore—relative the integral of the theorem—if $g \in G$ and h is a bounded, μ-measurable function, then $\int_X ghd\mu = \int_X ghd\nu$. We may therefore assume that μ itself is σ-finite.

From Lemma 5.2 there is an $n + 1$-dimensional space of bounded, continuous functions defined on X. From Lemma 5.1, one of these functions, q, satisfies the theorem. $\square$

6. Final comments, examples, and open questions

Example. There is a nonatomic, Baire measure space that is not σ-finite, but supports a continuous, bounded function with level sets of measure zero.

Proof. Let $X = \{(a,b) : 0 \leq a \leq 1; \quad 0 \leq b \leq 1\} \subseteq [0,1] \times [0,1]$. For each $a \in [0,1]$, let μ_a be one-dimensional Lebesgue measure on the vertical line segment $I_a = \{(a,x) : 0 \leq x \leq 1\}$.

Let X have the usual Euclidean 2-dimensional topology, and let Σ be the Baire sets. We define $\mu(U) = \sum_{0 \leq a \leq 1} \mu_a(U \cap I_a)$ for $U \in \Sigma$. Then (X, Σ, μ) is a nonatomic, Baire measure space that is not σ-finite, and $f(a,b) = b$ is a continuous, bounded function with level sets of measure zero. ☐

Problem. Does there exist a nonatomic Baire measure space (X, Σ, μ) such that every continuous function defined on X has a level set of measure greater than 0?

Problem. If (X, Σ, μ) is a σ-finite, nonatomic, Baire measure space, does there exist a continuous, bounded, integrable function with level sets of measure zero?

Comment. Lemma 5.1 is related to a 1957 theorem proved by I.C.Gohberg and M.G.Krein. They show that if E and F are n- and $n+1$-dimensional subspaces, respectively, of a normed linear space, then there is a member of F that has 0 as a best approximation from E. The relation between Lemma 5.1 and the Gohberg-Krein theorem can be seen in [14] and in G.G. Lorentz' book.

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Mathematics Department 0112, University of California, San Diego, La Jolla, California 92093

E-mail address: `dwulbert@ucsd.edu`