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CHERN-OSSERMAN INEQUALITY FOR MINIMAL SURFACES IN $\mathbf{H}^n$

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Abstract. We obtain Chern-Osserman's inequality of a complete properly immersed minimal surface in hyperbolic n-space, provided the L^2-norm of the second fundamental form of the surface is finite.

1. Introduction

Let M be a complete minimal surface in Euclidean space $\mathbf{R}^n$ with finite total Gaussian curvature. Then the total Gaussian curvature of M satisfies the Chern-Osserman inequality ([2], [6])

$$-\chi(M) \leq \frac{-1}{2\pi}\int_M K - k,$$

where K is the Gaussian curvature of M, $\chi(M)$ is the Euler characteristic of M and k is the number of ends of M. The explicit expression of the total Gaussian curvature was obtained by Jorge and Meeks:

$$\begin{aligned}-\chi(M) &= \frac{-1}{2\pi}\int_M K - \sup\frac{\text{area} M\cap B(t)}{\pi t^2}\\ &= \frac{1}{4\pi}\int_M |A|^2 - \sup\frac{\text{area} M\cap B(t)}{\pi t^2},\end{aligned}$$

where A is the second fundamental form of M and $B(t)$ is the extrinsic distance ball of radius t from a fixed point.

In the paper we present an analogue of the Chern-Osserman inequality of complete minimal surfaces in hyperbolic n-space $\mathbf{H}^n$ of constant curvature -1, namely

Theorem. *Let M be an oriented immersed complete minimal surface in $\mathbf{H}^n$, A the second fundamental form of M, r the distance of $\mathbf{H}^n$ from a fixed point and $M_t = \{x \in M : r(x) < t\}$. Suppose $\int_M |A|^2(x) < \infty$; then*

(1) $\sup \frac{area M_t}{\cosh t - 1} < +\infty$;

(2) $-\chi(M) \leq \frac{1}{4\pi}\int_M |A|^2 - \sup\frac{area(M_t)}{2\pi(\cosh t - 1)}$,

where $\chi(M)$ is the Euler characteristic of M. Consequently, M has finite topological type.

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The minimal surfaces in $\mathbf{H}^n$ have some properties similar to minimal surfaces in $\mathbf{R}^n$, such as the monotonicity formula (see Proposition 2.2 below). But there are many differences between minimal surfaces in $\mathbf{H}^n$ and those in $\mathbf{R}^n$, one of the differences is that the total Gaussian curvature of a complete minimal surface in $\mathbf{H}^n$ is always infinite (this can be seen from the correspondent Gauss equation). Another important difference comes from the fundamental work of M. T. Anderson [1] for the existence of minimal varieties in $\mathbf{H}^n$. By his results, minimal surfaces in the hyperbolic space are much richer than those in Euclidean space, and the asymptotic behavior of the surfaces is not "*regular*" in general; in particular, the Bernstein Theorem (a complete minimal graph in $\mathbf{R}^3$ is flat) does not hold in $\mathbf{H}^3$.

In [5] De Oliveira proved that if M is an immersed complete minimal surface in $\mathbf{H}^n$ with $\int_M |A|^2 < \infty$, then M is properly immersed and is conformally equivalent to a compact surface with a finite number of disks removed. M. Kokubu [4] established the Weierstrass type representation formula for minimal surfaces in hyperbolic space. By his result, the Gauss maps of minimal surfaces in hyperbolic space are neither holomorphic nor anti-holomorphic. So the method employed in [2] and [6] is not valid in our case.

2. Preliminaries

Let $\mathbf{H}^n$ be a hyperbolic n-space of constant curvature -1, and M a properly immersed complete minimal surface in $\mathbf{H}^n$. Denote the covariant derivative of $\mathbf{H}^n$ and M by D and ∇ respectively; the second fundamental form of M is defined by

$$\begin{aligned} &A: \ TM \otimes TM \to \ T^{\perp}M, \\ &A(X,Y) = D_XY - \nabla_XY \ , \ \text{for } X, Y \in C^{\infty}(TM). \end{aligned} \tag{2.1}$$

For a smooth function f on $\mathbf{H}^n$, and any two tangent vector fields $X, Y \in C^{\infty}(TM)$,

$$\begin{aligned} (D^2f)(X,Y) &= (Ddf)(X,Y) \\ &= X(df(Y)) - df(D_XY) \\ &= X(df(Y)) - df(\nabla_XY + A(X,Y)) \\ &= \nabla^2 f(X,Y) - \langle A(X,Y), \ Df\rangle, \end{aligned}$$

which, together with the fact that $(D^2r)(X,X) = \coth r(\langle X,X\rangle - \langle X, Dr\rangle^2)$, implies

Proposition 2.1. *For any unit tangent vector e of M,*

$$(\nabla^2 r)(e,e) = \coth r(1 - \langle e, \nabla r\rangle^2) + \langle A(e,e), \nabla^{\perp} r\rangle,$$

where $\nabla^{\perp} r$ is the projection of Dr onto the normal of M.

Let r be the distance function of H^n from a fixed point. By Sard's theorem, for almost all $t > 0$, $M_t = \{x \in M : \ r(x) < t\}$ is a related compact open subset of M with the boundary ∂M_t being a closed immersed curve of M. $\{M_t\}$ is a family of exhaustion of M. We will consider the growth of the area of M_t, and make use of the following notations for convenience:

$$v(t) = \text{area} M_t, \quad \text{and} \quad R(t) = \int_{M_t} |A|^2.$$

Proposition 2.2 ([1], Theorem 1)**.** $\frac{v(t)}{\cosh t - 1}$ *is monotone nondecreasing in t, i.e., $v'(t)\cosh t - v(t)\sinh t \geq v'(t)$.*

Proposition 2.3. *Suppose M is an oriented and properly immersed complete minimal surface in $\mathbf{H}^n$. Then for almost all $t > 0$,*

$$v(t) + \frac{1}{2}R(t) + 2\pi\chi(M_t) = v'(t)\coth t - \int_{\partial M_t} \langle A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}), \frac{\nabla^\perp r}{|\nabla r|}\rangle.$$

Proof. Denote the geodesic curvature of ∂M_t in M by k_g^t, and by K the Gaussian curvature of M. Then the Gauss-Bonnet formula reads

$$\int_{\partial M_t} k_g^t + \int_{M_t} K = 2\pi\chi(M_t), \tag{2.2}$$

where $\chi(M_t)$ is the Euler characteristic of M_t, i.e. $\chi(M_t) = 2 - 2g - k$ with g being the genus of M_t and k being the number of components of ∂M_t.

Substituting the Gauss equation $K = -1 - \frac{1}{2}|A|^2$ into (2.2) we have

$$v(t) + \frac{1}{2}R(t) + 2\pi\chi(M_t) = \int_{\partial M_t} k_g^t. \tag{2.3}$$

Suppose e is the unit tangent vector of ∂M_t. Since the normal of ∂M_t in M is $\frac{\nabla r}{|\nabla r|}$,

$$\begin{aligned} k_g^t &= -\langle \nabla_e e, \frac{\nabla r}{|\nabla r|}\rangle \\ &= \frac{1}{|\nabla r|}(\nabla^2 r)(e,e) \\ &= \frac{1}{|\nabla r|}\left(\coth r + \langle A(e,e), \nabla^\perp r\rangle\right), \end{aligned} \tag{2.4}$$

where the last equality follows by Proposition 2.1.

Substituting (2.4) into (2.3), and using the fact that $v'(t) = \int_{\partial M_t} \frac{1}{|\nabla r|}$, and that $A(e,e) + A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}) = 0$, we complete the proof.

Lemma 2.4. *For $t > s > 0$,*

$$\frac{\int_{M_t} \cosh r}{\cosh^2 t} - \frac{\int_{M_s} \cosh r}{\cosh^2 s} = \int_{M_t - M(s)} \frac{1 + |\nabla^\perp r|^2 \sinh^2 r}{\cosh^3 r}.$$

Proof. By the minimality of M and Proposition 2.1, we observe that

$$\Delta r = \coth r(2 - |\nabla r|^2),$$

where Δ is the Laplacian of M. This yields

$$\Delta \cosh r = 2\cosh r. \tag{2.5}$$

Integrating (2.2) over M_t, and using Green's formula,

$$2\int_{M_t} \cosh r = \int_{\partial M_t} |\nabla r| \sinh r. \tag{2.6}$$

By using the co-area formula ([7]), we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\int_{M_t}\cosh r}{\cosh^2 t}\right) &= \frac{1}{\cosh^3 t}\left(\cosh t\frac{d}{dt}\int_{M_t}\cosh r - 2\sinh t\int_{M_t}\cosh r\right)\\
&= \frac{1}{\cosh^3 t}\left(\cosh t\int_{\partial M_t}\frac{\cosh r}{|\nabla r|} - \sinh t\int_{\partial M_t}|\nabla r|\sinh r\right)\\
&= \frac{1}{\cosh^3 t}\int_{\partial M_t}\left(\frac{\cosh^2 r}{|\nabla r|} - \sinh^2 r|\nabla r|\right)\\
&= \frac{1}{\cosh^3 t}\int_{\partial M_t}\frac{1}{|\nabla r|}(1+\sinh^2 r|\nabla^{\perp} r|^2).
\end{aligned} \tag{2.4}$$

The proposition is then proved by integrating (2.6) from s to t and the co-area formula.

Lemma 2.5.

$$\int_0^t v'(s)\cosh s ds \geq \frac{\cosh t+1}{2}v(t).$$

Proof. By Proposition 2.2, $v'(t) \geq \frac{v(t)\sinh t}{\cosh t - 1}$, so we have

$$\begin{aligned}
\int_0^t v'(s)\cosh s ds &= v(t)\cosh t - \int_0^t v(s)\sinh s ds\\
&\geq v(t)\cosh t - \int_0^t v'(s)(\cosh s - 1)ds\\
&= v(t)(\cosh t + 1) - \int_0^t v'(s)\cosh s ds,
\end{aligned}$$

which proves the lemma.

3. Proof of the Theorem

Suppose M is as in the Theorem. By the result of De Oliveira [5], M is properly immersed.

(1) By Proposition 2.3 we have

$$\frac{v'(t)\cosh t - v(t)\sinh t}{\sinh t} = \frac{1}{2}R(t) + 2\pi\chi(M_t) + \int_{\partial M_t}\langle A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}), \frac{\nabla^{\perp} r}{|\nabla r|}\rangle; \tag{3.1}$$

hence

$$\frac{d}{dt}\frac{v(t)}{\cosh t} \leq \frac{\sinh t}{\cosh^2 t}\left(\frac{1}{2}R(t) + 2\pi\chi(M_t)\right) + \int_{\partial M_t}\frac{|A|}{|\nabla r|}\frac{|\nabla^{\perp} r|\sinh t}{\cosh^2 t}.$$

Since $\chi(M_t) \leq 1$, integrating the above inequality from 0 to t and by using the co-area formula, we have

$$\frac{v(t)}{\cosh t} \leq 2\int_0^t (\frac{1}{2}R(s) + 2\pi)e^{-s}ds + \int_{M_t}|A|\frac{|\nabla^{\perp} r|\sinh r}{\cosh^2 r}. \tag{3.2}$$

By using the Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{v(t)}{\cosh t} &\le C_1 + \left(\int_{M_t} \frac{|A|^2}{\cosh r}\right)^{\frac{1}{2}} \left(\int_{M_t} \frac{|\nabla^\perp r|^2 \sinh^2 r}{\cosh^3 r}\right)^{\frac{1}{2}} \\ &\le C_1 + C_2 \left(\frac{\int_{M_t} \cosh r}{\cosh^2 t}\right)^{\frac{1}{2}} \text{ (by Proposition 2.3)} \\ &\le C_1 + C_2 \left(\frac{v(t)}{\cosh t}\right)^{\frac{1}{2}},\end{aligned} \tag{3.3}$$

where C_1 and C_2 are two constants independent of t. By Proposition 2.2, $\frac{v(t)}{\cosh t}$ is monotone non-decreasing in t; therefore, either $\sup \frac{v(t)}{\cosh t} \le C_1^2$ or

$$\frac{v(t)}{\cosh t} \le \left(\frac{v(t)}{\cosh t}\right)^{\frac{1}{2}} + C_2 \left(\frac{v(t)}{\cosh t}\right)^{\frac{1}{2}}$$

when t is large enough. It follows that

$$\sup \frac{v(t)}{\cosh t} \le \max\{C_1^2, (1+C_2)^2\}.$$

This proves (1) of the Theorem.

(2) By the arithmetic geometric mean inequality, (3.1) implies

$$\begin{aligned}\frac{v'(t)\cosh t - v(t)\sinh t}{\sinh t} &\le \frac{1}{2}R(t) + 2\pi\chi(M_t) + \frac{1}{2}\int_{\partial M_t} \left(\frac{|A|^2}{|\nabla r|} + \frac{|\nabla^\perp r|^2}{|\nabla r|}\right) \\ &\le \frac{1}{2}R(t) + 2\pi\chi(M_t) + \frac{1}{2}R'(t) + \int_{\partial M_t} \frac{|\nabla^\perp r|^2}{|\nabla r|}.\end{aligned} \tag{3.4}$$

Since by Proposition 2.1, $\Delta \cosh r = 2\cosh r$, Green's formula gives

$$\int_{\partial M_t} |\nabla r| \sinh r = \int_{M_t} 2\cosh r;$$

then by the co-area formula we have

$$\begin{aligned}\int_{\partial M_t} \frac{|\nabla^\perp r|^2}{|\nabla r|} &= \int_{\partial M_t} \frac{1}{|\nabla r|} - |\nabla r| \\ &= v'(t) - \frac{1}{\sinh t}\int_{\partial M_t} |\nabla r| \sinh r \\ &= v'(t) - \frac{1}{\sinh t}\int_{M_t} 2\cosh r \\ &= v'(t) - \frac{2}{\sinh t}\int_0^t v'(s)\cosh s ds.\end{aligned} \tag{3.5}$$

By Lemma 2.5 we obtain

$$\int_{\partial M_t} \frac{|\nabla^\perp r|^2}{|\nabla r|} \le v'(t) - \frac{\cosh t + 1}{\sinh t} v(t). \tag{3.6}$$

Substituting (3.6) into (3.4) we have

$$\begin{aligned}\frac{v'(t)\cosh t - v(t)\sinh t}{\sinh t} \leq {} & \frac{1}{2}(R(t) + R'(t)) + 2\pi\chi(M_t) \\ & + \frac{v'(t)\sinh t - v(t)\cosh t - v(t)}{\sinh t}.\end{aligned}$$

This implies

$$\begin{aligned}-2\pi\chi(M_t) & \leq \frac{1}{2}(R(t) + R'(t)) + \frac{(v'(t) + v(t))(\sinh t - \cosh t) - v(t)}{\sinh t} \\ & \leq \frac{1}{2}(R(t) + R'(t)) - \frac{v(t)}{\sinh t}.\end{aligned} \tag{3.7}$$

Since $\int_0^\infty R'(t)dt < \infty$, there is a monotone increasing sequence $\{t_i\}$ diverging to infinity such that $R'(t_i) \to 0$ as $i \to \infty$. Taking $t = t_i$ in (3.7) and letting i tend to infinity, we prove the theorem.

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