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RELATIONS BETWEEN THE TAYLOR SPECTRUM AND THE XIA SPECTRUM

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(Communicated by David R. Larson)

Dedicated to Professor Jyunji Inoue on his sixtieth birthday

Abstract. Let $\mathbf{T} = (T_1, T_2, ..., T_n)$ be a doubly commuting n-tuple of p-hyponormal operators T_j with unitary operators U_j from the polar decompositions $T_j = U_j|T_j|$ $(j = 1, ..., n)$. Let $\mathbf{U} = (U_1, ..., U_n)$ and $A = |T_1| \cdots |T_n|$. In this paper, we will show relations between the Taylor spectrum $\sigma_T(\mathbf{T})$ and the Xia spectrum $\sigma_X(\mathbf{U}, A)$.

1. Introduction

In [12], D. Xia introduced a class of semi-hyponormal tuples and a notion of spectrum for such tuples. We call this spectrum the *Xia spectrum*. Xia proved Putnam's inequality for semi-hyponormal tuples. In [6], M. Chō and T. Huruya generalized Putnam's inequality to p-hyponormal tuples. Also, in [9], B. P. Duggal showed a very interesting inequality of doubly commuting n-tuples of p-hyponormal operators. In this paper, we show that the Xia spectrum of a doubly commuting n-tuple $\mathbf{T} = (T_1, ..., T_n)$ of p-hyponormal operators T_j with unitary operators U_j from the polar decompositions $T_j = U_j|T_j|$ $(j = 1, ..., n)$ essentially coincides with its Taylor spectrum.

Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, let $\sigma(T)$ be the spectrum of T. An operator $T \in B(\mathcal{H})$ is called p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. If $p = \frac{1}{2}$, then T is called semi-hyponormal. Let W be a unitary operator and $A \in B(\mathcal{H})$. If

$$\mathcal{S}_W^{\pm}(A) = \text{s - }\lim_{n \to \pm\infty} (W^{-n} A W^n)$$

exist, then the operators $\mathcal{S}_W^{\pm}(A)$ are called the polar symbols of A (with respect to W). Let $T = U|T|$ be the polar decomposition of T. If T is semi-hyponormal and U is unitary, then $\mathcal{S}_U^{\pm}(|T|)$ exist (cf. [13]). In [13], D. Xia proved the following theorem:

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Theorem A (Theorem IV.4.1 of [13])**.** *Let $T = U|T|$ be a semi-hyponormal operator with U unitary. Then*

$$\sigma(T) = \bigcup_{0 \le k \le 1} \sigma(T_{(k)}),$$

where $T_{(k)} = k\mathcal{S}_U^+(T) + (1-k)\mathcal{S}_U^-(T)$.

2. Generalized polar symbols

Throughout this paper let p be such that $0 < p < \frac{1}{2}$. Let $T = U|T|$ be a p-hyponormal operator with U unitary. Since $U|T|^{2p}$ is semi-hyponormal, there exist $\mathcal{S}_U^{\pm}(|T|^{2p})$. For $0 \le k \le 1$, we denote

$$T_k = U\{k\mathcal{S}_U^+(|T|^{2p}) + (1-k)\mathcal{S}_U^-(|T|^{2p})\}^{\frac{1}{2p}};$$

we call the operators T_k the generalized polar symbols of T. Note that if an operator $T = U|T|$ is a semi-hyponormal operator with U unitary, then $T_{(k)} = T_k$ for every $0 \le k \le 1$. It is easy to check that T_k is a normal operator for every $0 \le k \le 1$. For $T \in B(\mathcal{H})$, let $\sigma_{na}(T)$ denote the normal approximate point spectrum of T, i.e., the set of all complex numbers z which satisfy the following condition: there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \|(T-z)x_n\| = \lim_{n\to\infty} \|(T-z)^*x_n\| = 0.$$

If T is a normal operator, then $\sigma(T) = \sigma_{na}(T)$. In [12] this spectrum is called the joint approximate point spectrum, but we use this term for n-tuple of operators. The following theorem holds.

Theorem B (Lemma I.2.4 of [13])**.** *Let $T \in B(\mathcal{H})$ and let $T = U|T|$ be the polar decomposition of T. Let $r > 0$. Then $re^{i\theta} \in \sigma_{na}(T)$ if and only if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} \|(|T|-r)x_n\| = \lim_{n\to\infty} \|(U-e^{i\theta})x_n\| = 0.$$

Therefore, for a semi-hyponormal operator $T = U|T|$ with U unitary and a non-zero $re^{i\theta} \in \mathbf{C}$, it follows that $re^{i\theta} \in \sigma_{na}(T_{(k)})$ if $re^{i\theta} \in \sigma(T_{(k)})$, because each $T_{(k)}$ is a normal operator $(0 \le k \le 1)$*.*

The following result was proved in [5]. For the sake of completeness, we will give a simple proof.

Theorem 1 (Theorem of [5])**.** *Let $T = U|T|$ be a p-hyponormal operator with U unitary. Then*

$$\sigma(T) = \bigcup_{0 \le k \le 1} \sigma(T_k).$$

For the proof of this theorem, we need the following result.

Theorem C (Theorem 3 of [4])**.** *Let $T = U|T|$ be a p-hyponormal operator with U unitary. Then*

$$\sigma(U|T|^{2p}) = \{\ r^{2p}e^{i\theta} \mid re^{i\theta} \in \sigma(T)\ \}.$$

Proof of Theorem 1. Note that $\mathcal{S}_U^-(|T|^{2p}) \le |T|^{2p} \le \mathcal{S}_U^+(|T|^{2p})$ (cf. Th.II.2.7 of [13]). If $0 \in \sigma(T)$, then $0 \in \sigma(|T|)$ and hence $0 \in \sigma(T_0)$ ($T_0 = U\{\mathcal{S}_U^-(|T|^{2p})\}^{\frac{1}{2p}}$). Conversely, let $0 \in \bigcup_{0 \le k \le 1} \sigma(T_k)$. Since T_k is normal, we have $0 \in \sigma(\mathcal{S}_U^-(|T|^{2p}))$ and hence $0 \in \sigma(|T|)$ (cf. Th.II.1.5 of [12]). Therefore, we have $0 \in \sigma(T)$. Next we prove that, for a non-zero $z = re^{i\theta} \in \mathbf{C}$, $z \in \sigma(T)$ if and only if $z \in \bigcup_{0\le k\le 1} \sigma(T_k)$. Let $S = U|T|^{2p}$. Then S is semi-hyponormal and from Theorem C we have

$$\begin{aligned} z \in \sigma(T) \quad &\Longleftrightarrow \quad r^{2p}e^{i\theta} \in \sigma(S) \\ \Longleftrightarrow \quad \exists k\ (0 \le k \le 1)\ \ ;\ \ r^{2p}e^{i\theta} \in \sigma(S_{(k)}) \quad &\text{(from Theorem A)} \\ \Longleftrightarrow \quad \exists k\ (0 \le k \le 1)\ \ ;\ \ r^{2p}e^{i\theta} \in \sigma_{na}(S_{(k)}) \quad &\text{(from Theorem B)} \\ \Longleftrightarrow \quad \exists k\ (0 \le k \le 1)\ \ ;\ \ re^{i\theta} \in \sigma_{na}(T_k)& \\ \Longleftrightarrow \quad z \in \bigcup_{0\le k\le 1} \sigma(T_k).& \end{aligned}$$

The proof is now complete.

3. The Taylor spectrum and the Xia spectrum

For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n)$, the Taylor spectrum and the joint approximate point spectrum of $\mathbf{T}$ are denoted by $\sigma_T(\mathbf{T})$ and $\sigma_{ja}(\mathbf{T})$, respectively. It is well known that $\sigma_T(\mathbf{T}) = \sigma_{ja}(\mathbf{T})$ if $\mathbf{T}$ is a commuting n-tuple of normal operators. If $\mathbf{T} = (T_1, ..., T_n)$ is a doubly commuting n-tuple of p-hyponormal operators, then, by Theorem 7 of [3], it follows that $\sigma_T(\mathbf{T}) = \{(z_1, ..., z_n) \in \mathbf{C}^n : (\bar{z_1}, ..., \bar{z_n}) \in \sigma_{ja}(\mathbf{T}^*)\}$, where $\mathbf{T}^* = (T_1^*, ..., T_n^*)$. Let $\mathbf{U} = (U_1, ..., U_n)$ be a commuting n-tuple of unitary operators. Let $\mathbf{Q}_j$ $(j = 1, ..., n)$ on $B(\mathcal{H})$ be defined by

$$\mathbf{Q}_j A = A - U_j A U_j^* \quad (A \in B(\mathcal{H})).$$

Let $A \in B(\mathcal{H})$ and $A \ge 0$. An $(n+1)$-tuple $(\mathbf{U}, A)$ is called p-hyponormal if

$$\mathbf{Q}_{j_1} \cdots \mathbf{Q}_{j_m} A^{2p} \ge 0$$

for all $1 \le j_1 < \cdots < j_m \le n$. We simply denote $\mathcal{S}_{U_j}^{\pm}(A)$ by $\mathcal{S}_j^{\pm}(A)$ for every $j = 1, ..., n$. Let $(\mathbf{U}, A)$ be a p-hyponormal tuple and $0 \le k \le 1$. We denote

$$(k\mathcal{S}_j^+ + (1-k)\mathcal{S}_j^-)_p A = \{k\mathcal{S}_j^+(A^{2p}) + (1-k)\mathcal{S}_j^-(A^{2p})\}^{\frac{1}{2p}}.$$

For $\mathbf{k} = (k_1, ..., k_n) \in [0,1]^n$, the general polar symbols $A_{\mathbf{k}}$ of A are defined by

$$A_{\mathbf{k}} = \prod_{j=1}^{n} (k_j\mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-)_p A.$$

Then, by [6], $(\mathbf{U}, A_{\mathbf{k}})$ is a commuting $(n+1)$-tuple of normal operators for every $\mathbf{k} \in [0,1]^n$. We define the Xia spectrum $\sigma_X(\mathbf{U}, A)$ of $(\mathbf{U}, A)$ by

$$\sigma_X(\mathbf{U}, A) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, A_{\mathbf{k}}).$$

By Theorem 2 of [6] it follows that, for a p-hyponormal tuple $(\mathbf{U}, A)$,

$$\|\mathbf{Q}_1 \cdots \mathbf{Q}_n A^{2p}\| \ \le\ \frac{2p}{(2\pi)^n} \int \cdots \int_{\sigma_X(\mathbf{U},A)} r^{2p-1} d\theta_1 \cdots d\theta_n dr.$$

We now have the following

Lemma 2. *Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of p-hyponormal operators $T_j = U_j|T_j|$ with U_j unitary operators $(j = 1, ..., n)$, and let $\mathbf{U} = (U_1, ..., U_n)$ and $A = |T_1| \cdots |T_n|$. Then $(\mathbf{U}, A)$ is p-hyponormal.*

Proof. Since $A^{2p} = |T_1|^{2p} \cdots |T_n|^{2p}$, we have

$$\mathbf{Q}_j A^{2p} = (\prod_{i \neq j} |T_i|^{2p})(|T_j|^{2p} - U_j |T_j|^{2p} U_j^*)$$

for every j $(j = 1, ..., n)$. Hence $(\mathbf{U}, A)$ is p-hyponormal.

With the above notations (Lemma 2), we also have, using the above,

$$\| \prod_{j=1}^{n} (|T_j|^{2p} - |T_j^*|^{2p}) \| \leq \frac{2p}{(2\pi)^{2n}} \int \cdots \int_{\sigma_X(\mathbf{U},A)} r^{2p-1} d\theta_1 \cdots d\theta_n dr$$

(this inequality is due to Duggal [9]), and

$$(k_j \mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-)_p A = (\prod_{i \neq j} |T_i|) \{k_j \mathcal{S}_j^+ (|T_j|^{2p}) + (1-k_j) \mathcal{S}_j^- (|T_j|^{2p})\}^{\frac{1}{2p}}.$$

Hence, for every $\mathbf{k} = (k_1, ..., k_n) \in [0,1]^n$, it follows that

$$A_{\mathbf{k}} = \prod_{j=1}^{n} A_j,$$

where $A_j = \{k_j \mathcal{S}_j^+ (|T_j|^{2p}) + (1-k_j) \mathcal{S}_j^- (|T_j|^{2p})\}^{\frac{1}{2p}}$ $(j = 1, ..., n)$. We prove the following

Theorem 3. *Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of p-hyponormal operators with unitary operators U_j from the polar decompositions $T_j = U_j |T_j|$ $(j = 1, ..., n)$. Let $\mathbf{U} = (U_1, ..., U_n)$ and $A = |T_1| \cdots |T_n|$. If $(z_1, ..., z_n, a) \in \sigma_X(\mathbf{U}, A)$, then there exist non-negative numbers $a_1, ..., a_n$ such that $(z_1 a_1, ..., z_n a_n) \in \sigma_T(\mathbf{T})$ and $a = a_1 \cdots a_n$.*

Conversely, if $(z_1 a_1, ..., z_n a_n) \in \sigma_T(\mathbf{T})$, then $(z_1, ..., z_n, a_1 \cdots a_n) \in \sigma_X(\mathbf{U}, A)$, where $|z_j| = 1$ and $a_j \geq 0$ for every j $(j = 1, ..., n)$.

For the proof of this theorem, we need the following Berberian extension theorem.

Theorem D (Theorem 1 of [1])**.** *Let $B(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. Then there exist an extension space $\mathcal{K}$ of $\mathcal{H}$ and a faithful *-representation of $B(\mathcal{H})$ into $B(\mathcal{K}) : T \to T^\circ$ such that*

$$\sigma_{ja}(T_1, ..., T_n) = \sigma_{ja}(T_1^\circ, ..., T_n^\circ) = \sigma_p(T_1^\circ, ..., T_n^\circ),$$

where $\sigma_p(T_1, ..., T_n)$ is the joint point spectrum of $(T_1, ..., T_n)$. Moreover, if T is p-hyponormal, then T° is also p-hyponormal.

Proof of Theorem 3. First we assume that $(z_1, ..., z_n, a) \in \sigma_X(\mathbf{U}, A)$. We show by induction that there exist $a_1, ..., a_n$ $(\forall a_j \geq 0)$ such that

$$(z_1 a_1, ..., z_n a_n) \in \sigma_T(\mathbf{T}) \text{ and } a = a_1 \cdots a_n.$$

If $n = 1$, Theorem 3 holds by Theorem 3 of [6]. By inductive hypothesis, there exist $\mathbf{k} = (k_1, ..., k_n) \in [0,1]^n$ and a sequence $\{x_m\}$ of unit vectors such that

$$(U_j - z_j) x_m \to 0 \ (j = 1, ..., n) \text{ and } (A_{\mathbf{k}} - a) x_m \to 0,$$

where $A_{\mathbf{k}} = \prod_{j=1}^{n} (k_j \mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-)_p A$. By Lemma 2 we have

$$A_{\mathbf{k}} = \prod_{j=1}^{n} A_j,$$

where $A_j = \{k_j\mathcal{S}_j^+(|T_j|^{2p}) + (1-k_j)\mathcal{S}_j^-(|T_j|^{2p})\}^{\frac{1}{2p}}$. By Theorem D, let $\mathcal{K}$ be the extension space of $\mathcal{H}$. Then

$$\mathcal{M} \;=\; \mathrm{Ker}(U_1^\circ - z_1)\;\cap\cdots\cap\;\mathrm{Ker}(U_n^\circ - z_n)\;\cap\;\mathrm{Ker}(A_{\mathbf{k}}^\circ - a)$$

is a non-zero subspace of $\mathcal{K}$. Since $(U_1^\circ, ..., U_n^\circ, A_1^\circ, ..., A_n^\circ)$ is a commuting $2n$-tuple, $\mathcal{M}$ is an invariant subspace for $A_1^\circ, ..., A_n^\circ$. Also since $a \in \sigma(A^\circ_{\mathbf{k}|\mathcal{M}})$, there exist $a_1, ..., a_n$ and a non-zero vector $x^\circ \in \mathcal{M}$ such that

$$(A_j^\circ - a_j)x^\circ \;=\; 0 \;\text{ for every } j\ (j = 1, ..., n) \;\text{ and }\; a = a_1\cdots a_n,$$

by Theorem D and the spectral mapping theorem for the joint spectrum. Let

$$\mathcal{N} = \mathrm{Ker}(U_n^\circ - z_n)\;\cap\;\mathrm{Ker}(A_n^\circ - a_n).$$

Then

$$(z_1, ..., z_{n-1}, a_1\cdots a_{n-1}) \in \sigma_X(\mathbf{U}', \mathbf{A}'),$$

where $\mathbf{U}' = (U_1, ..., U_{n-1})$ and $\mathbf{A}' = \displaystyle\prod_{j=1}^{n-1} A_j$. By Theorem D and the inductive hypothesis, we have

$$(z_1a_1, ..., z_{n-1}a_{n-1}) \;\in \sigma_T(T_1, ..., T_{n-1}).$$

Since $\mathbf{S} = (T^\circ_{1|\mathcal{N}}, ..., T^\circ_{n-1|\mathcal{N}})$ is a doubly commuting $(n-1)$-tuple of p-hyponormal operators on $\mathcal{N}$ and $(z_1a_1, ..., z_{n-1}a_{n-1}) \in \sigma_T(\mathbf{S})$, Theorem 7 of [3] and Theorem D imply that there exists a non-zero vector y° in $\mathcal{N}$ such that

$$(T_j^\circ - z_ja_j)^*y^\circ \;=\; 0 \;\text{ for every } j\ (j = 1, ..., n-1).$$

Let

$$\mathcal{L} \;=\; \bigcap_{j=1}^{n-1} \mathrm{Ker}((T_j^\circ - z_ja_j)^*).$$

Then $\mathcal{N} \cap \mathcal{L}$ is a non-zero subspace of $\mathcal{K}$. Hence we have $(z_n, a_n) \in \sigma_{jp}(U^\circ_{n|\mathcal{L}}, A^\circ_{n|\mathcal{L}})$ and $(z_n, a_n) \in \sigma_X(U^\circ_{n|\mathcal{L}}, |T_{n|\mathcal{L}}|^\circ)$. Also by the induction we have

$$z_na_n \;\in\; \sigma(T^\circ_{n|\mathcal{L}}).$$

Since $T^\circ_{n|\mathcal{L}}$ is a p-hyponormal operator on $\mathcal{L}$, there exists a non-zero vector $w^\circ \in \mathcal{L}$ such that

$$(T_n^\circ - z_na_n)^*w^\circ \;=\; 0.$$

Therefore, there exists a sequence $\{x_m\}$ of unit vectors such that

$$(T_j - z_ja_j)^*x_m \;\to\; 0 \;\text{ for every } j\ (j = 1, ..., n).$$

Hence we have $(z_1a_1, ..., z_na_n) \;\in\; \sigma_T(\mathbf{T})$.

Conversely, we assume that $(z_1a_1, ..., z_na_n) \in \sigma_T(\mathbf{T})$. Also assume that the theorem holds for doubly commuting $(n-1)$-tuples of p-hyponormal operators. By Theorem 7 of [3] there exists a sequence $\{x_m\}$ of unit vectors such that

$$(T_j - z_ja_j)^*x_m \;\to\; 0 \;\text{ for every } j\ (j = 1, ..., n). \tag{1}$$

Consider the extension space $\mathcal{K}$ of $\mathcal{H}$ and let

$$\mathcal{U} \;=\; \mathrm{Ker}((T_n^\circ - z_na_n)^*).$$

By Theorem D and (1) there exists $z^\circ \in \mathcal{U}$ such that

$$(T_j^\circ - z_ja_j)^*z^\circ \;=\; 0 \;\text{ for every } j\ (j = 1, ..., n-1).$$

Since $(T_1^\circ, ..., T_{n-1}^\circ)$ is a commuting $(n-1)$-tuple of p-hyponormal operators on $\mathcal{U}$, it holds that $(z_1a_1, ..., z_{n-1}a_{n-1}) \in \sigma_T(T_{1|\mathcal{U}}^\circ, ..., T_{n-1|\mathcal{U}}^\circ)$. By the inductive hypothesis

$$(z_1, ..., z_{n-1}, a_1 \cdots a_{n-1}) \in \sigma_X(\mathbf{U}', \mathbf{A}'),$$

where $\mathbf{U}' = (U_{1|\mathcal{U}}^\circ, ..., U_{n-1|\mathcal{U}}^\circ)$ and $A' = |T_{1|\mathcal{U}}^\circ| \cdots |T_{n-1|\mathcal{U}}^\circ|$. Hence there exist $(m_1, ..., m_{n-1}) \in [0,1]^{n-1}$ and a non-zero vector $u^\circ \in \mathcal{U}$ such that

$$(U_j^\circ - z_j)u^\circ = (A_1^\circ \cdots A_{n-1}^\circ - a_1 \cdots a_{n-1})u^\circ = 0,$$

where $A_j = \{m_j \mathcal{S}_j^+(|T_j|^{2p}) + (1-m_j)\mathcal{S}_j^-(|T_j|^{2p})\}^{\frac{1}{2p}}$ for every j $(j = 1, ..., n-1)$. Next let

$$\mathcal{V} = \bigcap_{j=1}^{n-1} \operatorname{Ker}(U_j^\circ - z_j) \cap \operatorname{Ker}(A_1^\circ \cdots A_{n-1}^\circ - a_1 \cdots a_{n-1}).$$

Since $\mathcal{U} \cap \mathcal{V}$ is a non-zero subspace, we have

$$z_n a_n \in \sigma(T_{n|\mathcal{V}}^\circ).$$

Hence by Theorem 1 there exists $0 \le m_n \le 1$ such that $z_n a_n \in \sigma(U_n A_n)$, where $A_n = \{m_n \mathcal{S}_n^+(|T_n|^{2p}) + (1-m_n)\mathcal{S}_n^-(|T_n|^{2p})\}^{\frac{1}{2p}}$. Since $U_n A_n$ is a normal operator, by Theorem D there exists $v^\circ \in \mathcal{V}$ such that

$$(U_n^\circ - z_n)v^\circ = (A_n^\circ - a_n)v^\circ = 0.$$

Let $\mathbf{m} = (m_1, ..., m_n)$ and $A_{\mathbf{m}} = \prod_{j=1}^{n} A_j$. By Theorem D we have

$$(z_1, ..., z_n, a_1 \cdots a_n) \in \sigma_p(U_1^\circ, ..., U_n^\circ, A_{\mathbf{m}}^\circ),$$

and hence $(z_1, ..., z_n, a_1 \cdots a_n) \in \sigma_{ja}(\mathbf{U}, A_{\mathbf{m}})$. Using the definition of the Xia spectrum, we obtain

$$(z_1, ..., z_n, a_1 \cdots a_n) \in \sigma_X(\mathbf{U}, A).$$

The proof is now complete.

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