

LIIOUVILLE NUMBERS, RAJCHMAN MEASURES, AND SMALL CANTOR SETS

CHRISTIAN E. BLUHM

(Communicated by Christopher D. Sogge)

ABSTRACT. We show that the set of Liouville numbers carries a positive measure whose Fourier transform vanishes at infinity. The proof is based on a new construction of a Cantor set of Hausdorff dimension zero supporting such a measure.

1. INTRODUCTION

In the year 1844 JOSEPH LIIOUVILLE constructed an interesting class of transcendental numbers, namely

$$\mathbb{L} = \{x \in \mathbb{R} \setminus \mathbb{Q} : \forall n \in \mathbb{N} : \exists q \in \mathbb{N} : \|qx\| < q^{-n}\},$$

now called the set of *Liouville numbers*. Here $\|x\| = \min_{m \in \mathbb{Z}} |x - m|$ denotes the distance of a real number x to the nearest integer. For example, the number $\sum 10^{-k!} = 0.110001000\dots$ (where the 1 is only in places $n!$) belongs to \mathbb{L} . From the well known theorem of JARNIK [4] and BESICOVITCH [1] it follows immediately that \mathbb{L} has Hausdorff dimension zero, so we consider \mathbb{L} to be a ‘small’ set.

In this note we show that \mathbb{L} supports a positive measure whose Fourier transform vanishes at infinity. Such measures are called *Rajchman measures*; see the survey article by LYONS [6] for references. A detailed discussion of related constructions can be found in KÖRNER’s paper [5].

The proof of our result is based on a new construction of a Cantor set with Hausdorff dimension zero carrying a Rajchman measure.

2. MAIN RESULTS

In the sequel \mathbb{P}_M denotes the set of prime numbers between M and $2M$ where M is a positive integer. We choose a sequence of positive integers $(M_k)_{k \in \mathbb{N}}$ with $M_1 < 2M_1 < M_2 < 2M_2 < M_3 < 2M_3 < \dots$ and define the set

$$S_\infty = \bigcap_{k=1}^{\infty} \bigcup_{p \in \mathbb{P}_{M_k}} \{x \in [0, 1] : \|px\| \leq p^{-1-k}\}.$$

Received by the editors September 1, 1998 and, in revised form, October 19, 1998.
1991 *Mathematics Subject Classification*. Primary 42A38; Secondary 28A80.
Key words and phrases. Liouville numbers, Rajchman measure, Cantor set.

In fact, S_∞ is compact because $\overline{E}_k(p) = \{x \in [0, 1] : \|px\| \leq p^{-1-k}\}$ equals

$$(2.1) \quad [0, p^{-2-k}] \cup \bigcup_{m=1}^{p-1} \left[\frac{m}{p} - p^{-2-k}, \frac{m}{p} + p^{-2-k} \right] \cup [1 - p^{-2-k}, 1].$$

Proposition 2.1. S_∞ is a Cantor set of Hausdorff dimension zero.

Proof. According to (2.1) the set $\overline{E}_k(p)$ can be covered by $p + 1$ intervals of length $\leq 2p^{-2-k}$. For every $k \in \mathbb{N}$ and $\alpha = 3/(2 + k)$ this implies

$$H^\alpha(S_\infty) \leq \sum_{p \in \mathbb{P}_{M_k}} (p + 1) (2p^{-2-k})^{3/(2+k)} < \infty.$$

Therefore, S_∞ has Hausdorff dimension zero. □

For the following theorem recall that the Fourier transform of a positive bounded measure μ is defined by

$$\hat{\mu}(x) = \int_{\mathbb{R}} e^{-2\pi ixt} d\mu(t) \quad (x \in \mathbb{R}).$$

Theorem 2.2. There exists a sequence $(M_k)_{k \in \mathbb{N}}$ such that the corresponding set S_∞ supports a positive measure μ_∞ with

$$\lim_{|x| \rightarrow \infty} \hat{\mu}_\infty(x) = 0.$$

As an application we obtain

Theorem 2.3. The set $S_\infty \setminus \mathbb{Q}$ is contained in \mathbb{L} . Therefore, the set of Liouville numbers carries a Rajchman measure.

Proof. From the definition of \mathbb{L} and S_∞ it is obvious that $S_\infty \setminus \mathbb{Q} \subset \mathbb{L}$. The Rajchman measure μ_∞ is supported by S_∞ . Removing the rational points from S_∞ means removing a zero set from the support of μ_∞ . By a simple regularity argument we can replace $\text{supp}(\mu_\infty) \setminus \mathbb{Q}$ by a compact set with positive μ_∞ -measure. □

3. PROOF OF THEOREM 2.2

Proof. The proof of Theorem 2.2 is based on a modification of a construction elaborated in [2]. There (in section 3) we constructed, for given positive $\alpha > 0$ and positive integers M , certain 1-periodic functions $g_M \in C^2(\mathbb{R})$ with

$$(3.1) \quad \text{supp}(g_M) \subseteq \bigcup_{p \in \mathbb{P}_M} \{x : \|px\| \leq p^{-1-\alpha}\} \quad \text{and} \quad \hat{g}_M(0) = 1.$$

Moreover, these functions g_M had the following nice property ([2], Lemma 3.2):

Lemma 3.1. For every $\psi \in C_c^2(\mathbb{R})$ and $\delta > 0$ there exists $M_0 = M_0(\psi, \delta)$ s.t.

$$\left| [\psi g_M]^\wedge(x) - \hat{\psi}(x) \right| \leq \delta \cdot \theta(x) \quad \forall x \in \mathbb{R}$$

for all $M \geq M_0$, where $\theta(x) = (1 + |x|)^{-1/(2+\alpha)} \cdot \log(e + |x|) \cdot \log(e + \log(e + |x|))$.

We will apply Lemma 3.1 to our situation by replacing α by k and θ by

$$\theta_k(x) = (1 + |x|)^{-1/(2+k)} \cdot \log(e + |x|) \cdot \log(e + \log(e + |x|)) \quad (k \in \mathbb{N}).$$

To be more precise, we fix an initial function $\psi_0 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with

$$\psi_0 \in C_c^2(\mathbb{R}), \quad \int \psi_0(x)dx = 1, \quad \psi_0|_{]0,1[} > 0, \quad \text{and} \quad \psi_0|_{\mathbb{R} \setminus [0,1]} \equiv 0.$$

Next we define a sequence $(\tau_k)_{k \in \mathbb{N}}$ by $\tau_k = (\max_{x \in \mathbb{R}} \theta_k(x))^{-1}$. As a first step we replace α in the setting above by $k = 1$. According to Lemma 3.1 we can find a positive integer $M_1 = M_1(\psi_0, \tau_1 3^{-1})$ such that

$$\left| [\psi g_{M_1}]^\wedge(x) - \hat{\psi}(x) \right| \leq \tau_1 3^{-1} \theta_1(x) \quad \forall x \in \mathbb{R}.$$

Now we repeat the same procedure, but this time replacing α by $k = 2$, θ_1 by θ_2 , and ψ by $\psi_0 g_{M_1}$. Then again Lemma 3.1 implies the existence of an integer $M_2 = M_2(\psi_0 g_{M_1}, \tau_2 3^{-2})$ such that

$$\left| [\psi g_{M_1} g_{M_2}]^\wedge(x) - [\psi_0 g_{M_1}]^\wedge(x) \right| \leq \tau_2 3^{-2} \theta_2(x) \quad \forall x \in \mathbb{R}.$$

By repeating this process we obtain for every index k an integer

$$M_k = M_k(\psi_0 g_{M_1} g_{M_2} \cdots g_{M_{k-1}}, \tau_k 3^{-k}) \quad (k \in \mathbb{N}),$$

fulfilling the corresponding estimation.

Now we assume S_∞ to be constructed according to $(M_k)_{k \in \mathbb{N}}$. We set

$$G_0 = 1, \quad \text{and} \quad G_k = g_{M_1} \cdots g_{M_k} \quad (k \in \mathbb{N}).$$

By Lemma 3.1 we obtain for every $k \in \mathbb{N}_0$ and all $x \in \mathbb{R}$

$$(3.2) \quad \left| [\psi_0 G_{k+1}]^\wedge(x) - [\psi_0 G_k]^\wedge(x) \right| \leq \tau_{k+1} 3^{-k-1} \theta_{k+1}(x).$$

Let λ be Lebesgue measure and define a sequence of measures by

$$\mu_k = \psi_0 G_k \lambda \quad (k \in \mathbb{N}_0).$$

Because of (3.2) the sequence $(\hat{\mu}_k)_{k \in \mathbb{N}_0}$ is a Cauchy sequence w.r.t. the supremum norm. Taking $\hat{g}_{M_k}(0) = 1$ into account (see (3.1)), we conclude the weak convergence of $(\mu_k)_k$ to a bounded measure μ_∞ (Lévy’s continuity theorem). Moreover, by (3.2) and a geometric series estimate we get $|\hat{\mu}_\infty(0) - \hat{\psi}(0)| \leq \frac{1}{2}$, so that μ_∞ has at least mass $\frac{1}{2}$. The claimed Fourier asymptotic of μ_∞ follows easily from (3.2) and a simple geometric series argument, also taking into account that $\hat{\mu}_p(x) = O(|x|^{-2})$ for fixed p . The construction of μ_∞ is based on successive multiplication of densities g_{M_k} . Therefore, by (3.1) it is clear that the support of μ_∞ must be contained in the Cantor set S_∞ . This concludes the proof of Theorem 2.2. □

Remark 3.2. Why prime numbers? Let us sketch the answer. The proof of Lemma 3.1 ([2], section 4) rests on the prime number theorem $\#\mathbb{P}_M \sim M/\log M$ (HARDY AND WRIGHT [3] (22.19.3)). So it is clear that although the number of primes between M and $2M$ is strictly increasing with M , the primes are somehow ‘thinning out’ at infinity. This observation is of great importance in the proof of Lemma 3.1, when one tries to allow the ‘ δ ’ in the estimation to become arbitrarily small.

REFERENCES

[1] Besicovitch, A. S., *Sets of fractional dimensions (IV): on rational approximation to real numbers*, J. Lond. Math. Soc. 9 (1934), 126-131
 [2] Bluhm, C., *On a theorem of Kaufman: Cantor-type construction of linear fractal Salem sets*, Ark. Mat. 36 (1998), 307-316 MR 99i:43009
 [3] Hardy, G. H., Wright, E. M., *An introduction to the theory of numbers*, Oxford University Press, 4th ed. (1971) MR 81i:10002 (5th edition)

- [4] Jarnik, V., *Zur metrischen Theorie der diophantischen Approximation*, Prace Mat.-Fiz. 36 (1928/29), 91-106
- [5] Körner, T. W., *On the theorem of Ivashev-Musatov III*, Proc. London Math. Soc. (3) 53 (1986), 143-192 MR **88f**:42021
- [6] Lyons, R., *Seventy Years of Rajchman Measures*, J. Fourier Anal. Appl., Kahane Special Issue (1995), 363-377 MR **97b**:42019

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GREIFSWALD, JAHNSTRASSE 15A, D-17487
GREIFSWALD, GERMANY

E-mail address: `bluhm@rz.uni-greifswald.de`