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CARDINAL SPLINE INTERPOLATION FROM $H^1(\mathbb{Z})$ TO $L_1(\mathbb{R})$

FANG GENSUN

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Abstract. Let $H^1(\mathbb{Z})$ be the discrete Hardy space, consisting of those sequences $y = \{y_j\}_{j\in\mathbb{Z}} \in l_p(\mathbb{Z})$, such that $Hy = \{Hy_j\} \in l_1(\mathbb{Z})$, where $Hy_j = \sum\limits_{k\neq j}(k-j)^{-1}y_j$, $j \in \mathbb{Z}$, is the discrete Hilbert transform of y. For a sequence $y = \{y_j\} \in l_1(\mathbb{Z})$, let $\mathcal{L}_m y(x) \in L_p(\mathbb{R})$ be the unique cardinal spline of degree $m-1$ interpolating to y at the integers. The norm of this operator, $\|\mathcal{L}_m\|_1 = \sup\{\|\mathcal{L}_m y\|_{L(\mathbb{R})}/\|y\|_{l(\mathbb{Z})}\}$, is called a Lebesgue constant from $l_1(\mathbb{Z})$ to $L_1(\mathbb{R})$, and it was proved that $\sup\limits_m\{\|\mathcal{L}_m\|_1\} = \infty$.

It is proved in this paper that

$$\sup_m\left\{\|\mathcal{L}_m y\|_{1(\mathbb{R})}/(\|y\|_{l(\mathbb{Z})} + \|\{H(-1)^j y_j\}\|_{l(\mathbb{Z})})\right\} \leqslant \left(1+\frac{\pi}{2}\right)\left(1+\frac{\pi}{3}\right).$$

1. Introduction

Denote the classical Lebesgue space on $\mathbb{R}$ by $L_p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, and let $\|\cdot\|_{p(\mathbb{R})}$ denote its norm.

For a natural number m, the space $\mathcal{S}_{m,p}(\mathbb{R}) = \{s\}$ of cardinal splines of degree $m-1$ is taken to consist of those functions satisfying:
(i) $s \in C^{m-2}(\mathbb{R})$,
(ii) $\|s\|_{p(\mathbb{R})} < \infty$, $1 \leqslant p \leqslant \infty$,
(iii) s reduces to a polynomial of degree at most $m-1$ on each of the intervals $[\nu + m/2, \nu + m/2 + 1]$, $\nu \in \mathbb{Z}$.

For a sequence $y = \{y_j\}_{j\in\mathbb{Z}} \in l_p(\mathbb{Z})$, we define the space of double infinite bounded sequences with the usual norm as follows:

$$\begin{aligned}
&\|\{y_j\}\|_{l_p(\mathbb{Z})} = \Big(\sum_{j\in\mathbb{Z}} |y_j|^p\Big)^{1/p}, \qquad 1 < p < \infty,\\
&\|\{y_j\}\|_{l(\mathbb{Z})} := \|\{y_j\}\|_{l_1(\mathbb{Z})},\\
&\|\{y_j\}\|_{l_\infty(\mathbb{Z})} = \sup_{j\in\mathbb{Z}}\{|y_j|\}.
\end{aligned}$$

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Schoenberg [9] proved that there is a unique element $\mathcal{L}_m y \in \mathcal{S}_{m,p}(\mathbb{R})$ interpolating the given data at integers, i.e.,

$$\mathcal{L}_m y(j) = y_j, \qquad j \in \mathbb{Z}. \tag{1.1}$$

The operator $\mathcal{L}_m : \ l_p(\mathbb{Z}) \longrightarrow \mathcal{S}_{m,p}(\mathbb{R})$ is called the cardinal spline interpolation operator of order m from $l_p(\mathbb{Z})$ to $L_p(\mathbb{R})$ and its norm

$$\|\mathcal{L}_m\|_p = \sup\{\|\mathcal{L}_m y\|_{p(\mathbb{R})} : \ \|y\|_{l_p(\mathbb{Z})} \leqslant 1\} \tag{1.2}$$

is referred to as the mth Lebesgue constant for cardinal spline interpolation. These numbers were investigated previously by many authors (see [6]–[8]).

Theorem A ([6])**.** *Let* $1 < p < \infty$*. Then*

$$\|\mathcal{L}_m\|_p \leqslant C_p, \tag{1.3}$$

where the constant C_p *is independent of* m*.*

Theorem B ([6])**.** *The norms of the* m*th order cardinal spline interpolation operators from* $l_1(\mathbb{Z})$ *to* $L_1(\mathbb{R})$ *satisfy*

$$\lim_{m\to\infty} \big(\|\mathcal{L}_m\|_1 - (4/\pi^2)\log m\big) = \big(2A/\pi\big) + 4/\pi^2\big[\log(4/\pi) + \gamma\big], \tag{1.4}$$

where γ *is the Euler–Mascheroni constant and*

$$A = \int_0^\pi t^{-1}\Big(\tan(\frac{t}{2}) - \frac{2}{\pi(\pi - t)}\Big)\,dt. \tag{1.5}$$

From Theorem B, we know that $\sup\limits_m\{\|\mathcal{L}_m\|_1\} = \infty$.

Let $H^1(\mathbb{Z})$ be the discrete Hardy space, consisting of those double infinite bounded sequences $y = \{y_j\} \in l_1(\mathbb{Z})$, such that $Hy = \{Hy_j\} \in l_1(\mathbb{Z})$, where

$$Hy_j = \sum_{k\neq j} \frac{y_j}{k - j}, \qquad j \in \mathbb{Z}, \tag{1.6}$$

is the discrete Hilbert transform of y. Thus $H^1(\mathbb{Z})$ is the subspace of $l_1(\mathbb{Z})$ consisting of those sequences $y = \{y_j\}$ for which the discrete Hilbert transform also belongs to $l_1(\mathbb{Z})$. Clearly

$$\|\{y_j\}\|_{H^1(\mathbb{Z})} := \|\{y_j\}\|_{l(\mathbb{Z})} + \|\{Hy_j\}\|_{l(\mathbb{Z})} \tag{1.7}$$

is a norm of $H^1(\mathbb{Z})$.

$H^1(\mathbb{Z})$ was introduced by Coifman and Weiss [3, p. 622] as an important example of the Hardy space $H^1(\mathbf{X})$, associated with a space $\mathbf{X}$ of homogeneous type, in order to extend the atomic decomposition theory for the classical Hardy spaces to a more general setting. It is well known that the Hardy space $H^1(\mathbb{R})$ is a proper closed subspace of $L_1(\mathbb{R})$, and many results in harmonic analysis and approximation theory are valid on $H^1(\mathbb{R})$ but are not correct on $L_1(\mathbb{R})$. We have found the same situation exists with respect to the Lebesgue constant of the cardinal spline interpolation operator.

Our main result is the following:

Theorem 1. *Let* $\{(-1)^j y_j\} \in H^1(\mathbb{Z})$*. Then for all* $m \in \mathbb{N}$

$$\|\mathcal{L}_m y\|_{L(\mathbb{R})} \leqslant \Big(1 + \frac{\pi}{2}\Big)\Big(1 + \frac{\pi}{3}\Big)\big\|\{(-1)^j y_j\}\big\|_{H^1(\mathbb{Z})}. \tag{1.8}$$

2. Interpolation operator of cardinal spline

Let $j(x)$ be the unique integer satisfying $j(x)-\frac{1}{2}\leqslant x<j(x)+\frac{1}{2}$, and let

$$\widetilde{H}y(x)=\sum{}' y_j(x-j)^{-1}, \tag{2.1}$$

where the sum $\sum'$ is taken over those $j\in\mathbb{Z}$ for which $j\neq j(x)$, and $\widetilde{H}y$ is named the mixed Hilbert transform of the sequence $y=\{y_j\}$. Following some ideas of [6], we have

Lemma 1. *Let $y\in H^1(\mathbb{Z})$. Then*

$$\|\widetilde{H}y\|_{1(\mathbb{R})}\leqslant\frac{\pi^2}{3}\|\{y_j\}\|_{H^1(\mathbb{Z})}. \tag{2.2}$$

Proof. From the definition of $j(x)$, we have $|j(x)-x|\leqslant\frac{1}{2}$, and for $j\neq j(x)$, we get

$$|j(x)-j|\leqslant|j(x)-x|+|x-j|\leqslant\frac{1}{2}+|x-j|;$$

therefore

$$\Big|\frac{j(x)-j}{x-j}\Big|\leqslant 1+\frac{1}{2}\frac{1}{|x-j|}\leqslant 2. \tag{2.3}$$

For $j\neq j(x)$,

$$\frac{1}{x-j}=\frac{1}{j(x)-j}+\frac{(j(x)-x)(j(x)-j)}{x-j}(j(x)-j)^{-2}.$$

Hence

$$\Big|\sum{}'\frac{y_j}{x-j}\Big|\leqslant\Big|\sum{}'\frac{y_j}{j(x)-j}\Big|+\sum{}'\frac{|y_j|}{|j(x)-j|^2}, \tag{2.4}$$

from which we obtain

$$\begin{aligned}\Big\|\sum{}'\frac{y_j}{x-j}\Big\|_{1(\mathbb{R})}&\leqslant\int_{\mathbb{R}}\Big(\Big|\sum{}'\frac{y_j}{j(x)-j}\Big|+\sum{}'\frac{|y_j|}{|j(x)-j|^2}\Big)\,dx\\&=\sum_{k\in\mathbb{Z}}\int_{k-\frac{1}{2}}^{k+\frac{1}{2}}\Big(\Big|\sum_{j\neq k}\frac{y_j}{k-j}\Big|+\sum_{j\neq k}\frac{|y_j|}{|k-j|^2}\Big)\,dx\\&=\sum_{k\in\mathbb{Z}}\Big|\sum_{j\neq k}\frac{y_j}{k-j}\Big|+\sum_{k\in\mathbb{Z}}\sum_{j\neq k}\frac{|y_j|}{|k-j|^2}\\&=\sum_{k\in\mathbb{Z}}\Big|\sum_{j\neq k}\frac{y_j}{k-j}\Big|+\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{k^2}\sum_{j\in\mathbb{Z}}|y_j|\\&\leqslant\|\{Hy_j\}\|_{l(\mathbb{Z})}+\frac{1}{3}\pi^2\|\{y_j\}\|_{l(\mathbb{Z})}\\&\leqslant\frac{1}{3}\pi^2\|\{y_j\}\|_{H^1(\mathbb{Z})},\end{aligned} \tag{2.5}$$

which completes the proof of Lemma 1. □

Let $W_\sigma y(x)=\sum\limits_{k\in\mathbb{Z}}y_k\operatorname{sinc}\sigma(x-k\pi/\sigma)$, and let

$$\|W_\sigma\|_1=\sup\big\{\|W_\sigma y(x)\|_{1(\mathbb{R})}:\ \big\|\{(-1)^jy_j\}\big\|_{H^1(\mathbb{Z})}\leqslant 1\big\},$$

where $\operatorname{sinc}x:=x^{-1}\sin x$ for $x\neq 0$ and 1 for $x=0$, W_σ is the well-known Whittaker operator and $W_\sigma y$ is the Whittaker cardinal series. From Lemma 1, we have

Theorem 2. *Let $\sigma > 0$. Then*

$$\|W_\sigma\|_1 \leqslant \Big(1+\frac{1}{3}\pi\Big)\Big(\frac{\pi}{\sigma}\Big). \tag{2.6}$$

Proof. We first consider the case $\sigma = \pi$:

$$\begin{aligned}|W_\pi y(x)| &= \Big|\sum_{j\in\mathbb{Z}} y_j \operatorname{sinc} \pi(x-j)\Big| \\ &\leqslant \Big|\frac{\sin \pi x}{\pi}\sum_{j\neq j(x)}(-1)^j\frac{y_j}{x-j}\Big| + |y_{j(x)} \operatorname{sinc} \pi(x-j(x))| \\ &\leqslant \frac{1}{\pi}\Big|\sum_{j\neq j(x)}(-1)^j\frac{y_j}{x-j}\Big| + |y_{j(x)}|.\end{aligned}$$

Therefore, it follows from Lemma 1 that we have

$$\begin{aligned}\|W_\pi y(x)\|_{L(\mathbb{R})} &\leqslant \frac{\pi}{3}\|\{(-1)^j y_j\}\|_{H^1(\mathbb{Z})} + \|\{y_j\}\|_{l(\mathbb{Z})} \\ &\leqslant \Big(1+\frac{\pi}{3}\Big)\|\{(-1)^j y_j\}\|_{H^1(\mathbb{Z})}.\end{aligned} \tag{2.7}$$

By changing scale, we obtain from (2.7) that

$$\|W_\sigma\|_1 \leqslant \Big(1+\frac{\pi}{3}\Big)\Big(\frac{\pi}{\sigma}\Big).$$

□

Denote by $L_p^m(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, $m \in \mathbb{N}$, the subspace of f in $L_p(\mathbb{R})$ for which the $(m-1)$th derivative of f exists and is locally absolutely continuous on $\mathbb{R}$, and for which $\|f^{(m)}\|_{p(\mathbb{R})}$ is finite. By [4], if $f \in L_p^m(\mathbb{R})$, then

$$\|\{f(j)\}\|_{p(\mathbb{Z})} \leqslant \|f\|_{p(\mathbb{R})} + \|f'\|_{p(\mathbb{R})} < \infty;$$

therefore, it follows from Schoenberg [9] that for every $f \in L_p^m(\mathbb{R})$, there is a unique $\mathcal{L}_m f \in \mathcal{S}_{m,p}(\mathbb{R})$, such that $\mathcal{L}_m f(j) = f(j)$ for all $j \in \mathbb{Z}$. Moreover, we have

Lemma 2 ([5])**.** *Let $f \in L_1^m(\mathbb{R})$, $m \in \mathbb{N}$, and let $\mathcal{L}_m f$ be the unique cardinal spline of degree $m-1$ interpolating to $\{f(j)\}_{j\in\mathbb{Z}}$ at the integers. Then*

$$\|f - \mathcal{L}_m f\|_{1(\mathbb{R})} \leqslant \frac{\mathcal{K}_m}{\pi^m}\|f^{(m)}\|_{1(\mathbb{R})}, \tag{2.8}$$

where $\mathcal{K}_m$ is the Favard constant,

$$\mathcal{K}_m := \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k(m+1)}}{(2k+1)^{m+1}}, \tag{2.9}$$

and

$$1 = \mathcal{K}_0 < \mathcal{K}_2 < \cdots < \frac{4}{\pi} < \cdots < \mathcal{K}_3 < \mathcal{K}_1 = \frac{\pi}{2}.$$

Remark 1. de Boor and Schoenberg [2] proved that equation (2.8) is also valid for m even and $p = \infty$.

Let $\mathcal{E}_\sigma(\mathbb{R})$, $\sigma > 0$, be the restriction on $\mathbb{R}$ of entire functions of exponential type σ, and let

$$B_{\sigma,p} = \mathcal{E}_\sigma(\mathbb{R}) \cap L_p(\mathbb{R}), \quad 1 \leqslant p \leqslant \infty, \qquad B_\sigma := B_{\sigma,\infty}.$$

It is well known that $B_{\sigma,p} \subseteq B_{\sigma,q}$, $1 \leqslant p < q \leqslant \infty$.

Lemma 3 ([1, p. 211] Inequality of Bernstein's type)**.** *Let* $f \in B_{\sigma,p}$, $1 \leqslant p \leqslant \infty$, $\sigma > 0$. *Then*

$$\|f'\|_{p(\mathbb{R})} \leqslant \sigma \|f\|_{p(\mathbb{R})}.$$

Lemma 4 ([10])**.** *Let* $y = \{y_j\} \in l_2$. *Then there is a unique* $f \in B_{\pi,2}$, *interpolating the given data* $y = \{y_j\}_{j\in\mathbb{Z}}$ *at the integers, and* f *is represented by*

$$f(x) = \sum_{j\in\mathbb{Z}} y_j \operatorname{sinc} \pi(x-j), \quad \text{for all } x \in \mathbb{R},$$

and the series $\sum\limits_{j\in\mathbb{Z}} y_j \operatorname{sinc} \pi(x-j)$ *converges uniformly on* $\mathbb{R}$.

Proof of Theorem 1. Let $\{(-1)^j y_j\} \in H^1(\mathbb{Z})$. Then $\{y_j\} \in l_2(\mathbb{Z})$. By Lemma 4, there exists a function $f \in B_{\pi,2}$ such that $f(j) = y_j$ for all $j \in \mathbb{Z}$, hence $f \in B_\pi$. It follows from Theorem 2 that $f \in L_1(\mathbb{R})$; therefore $f \in B_{\pi,1}$. Using Lemma 2 and Bernstein's inequality we get

$$\begin{aligned} \|f - \mathcal{L}_m f\|_{1(\mathbb{R})} &\leqslant \frac{\mathcal{K}_m}{\pi^m} \|f^{(m)}\|_{1(\mathbb{R})} \\ &\leqslant \mathcal{K}_m \|f\|_{1(\mathbb{R})}, \end{aligned} \tag{2.10}$$

which together with (2.9) and Theorem 2 gives

$$\begin{aligned} \|\mathcal{L}_m y\|_{1(\mathbb{R})} = \|\mathcal{L}_m f\|_{1(\mathbb{R})} &\leqslant (1 + \mathcal{K}_m)\|f\|_{1(\mathbb{R})} \\ &\leqslant \left(1 + \frac{\pi}{2}\right)\left(1 + \frac{\pi}{3}\right)\|\{(-1)^j y_j\}\|_{H^1(\mathbb{Z})}, \end{aligned}$$

which completes the proof of Theorem 1. □

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DEPARTMENT OF MATHEMATICS, BEIJING NORMAL UNIVERSITY, BEIJING, 100875, PEOPLE'S REPUBLIC OF CHINA

E-mail address: `fanggs@ns.bnu.edu.cn`