PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 9, Pages 2597–2601
S 0002-9939(00)05290-4
Article electronically published on February 21, 2000

CARDINAL SPLINE INTERPOLATION FROM $H^1(\mathbb{Z})$ TO $L_1(\mathbb{R})$

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(Communicated by J. Marshall Ash)

Abstract. Let $H^1(\mathbb{Z})$ be the discrete Hardy space, consisting of those sequences $y = \{y_j\}_{j\in\mathbb{Z}} \in l_p(\mathbb{Z})$, such that $Hy = \{Hy_j\} \in l_1(\mathbb{Z})$, where $Hy_j = \sum_{k\neq j} (k-j)^{-1} y_j$, $j \in \mathbb{Z}$, is the discrete Hilbert transform of y. For a sequence $y = \{y_j\} \in l_1(\mathbb{Z})$, let $\mathcal{L}_m y(x) \in L_p(\mathbb{R})$ be the unique cardinal spline of degree $m-1$ interpolating to y at the integers. The norm of this operator, $\|\mathcal{L}_m\|_1 = \sup\{\|\mathcal{L}_m y\|_{L(\mathbb{R})}/\|y\|_{l(\mathbb{Z})}\}$, is called a Lebesgue constant from $l_1(\mathbb{Z})$ to $L_1(\mathbb{R})$, and it was proved that $\sup_m \{\|\mathcal{L}_m\|_1\} = \infty$.

It is proved in this paper that

$$\sup_m \left\{\|\mathcal{L}_m y\|_{1(\mathbb{R})}/(\|y\|_{l(\mathbb{Z})} + \|\{H(-1)^j y_j\}\|_{l(\mathbb{Z})})\right\} \leqslant \left(1+\frac{\pi}{2}\right)\left(1+\frac{\pi}{3}\right).$$

1. Introduction

Denote the classical Lebesgue space on $\mathbb{R}$ by $L_p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, and let $\|\cdot\|_{p(\mathbb{R})}$ denote its norm.

For a natural number m, the space $\mathcal{S}_{m,p}(\mathbb{R}) = \{s\}$ of cardinal splines of degree $m-1$ is taken to consist of those functions satisfying:

(i) $s \in C^{m-2}(\mathbb{R})$,

(ii) $\|s\|_{p(\mathbb{R})} < \infty$, $1 \leqslant p \leqslant \infty$,

(iii) s reduces to a polynomial of degree at most $m-1$ on each of the intervals $[\nu + m/2, \nu + m/2 + 1]$, $\nu \in \mathbb{Z}$.

For a sequence $y = \{y_j\}_{j\in\mathbb{Z}} \in l_p(\mathbb{Z})$, we define the space of double infinite bounded sequences with the usual norm as follows:

$$\|\{y_j\}\|_{l_p(\mathbb{Z})} = \Big(\sum_{j\in\mathbb{Z}} |y_j|^p\Big)^{1/p}, \qquad 1 < p < \infty,$$
$$\|\{y_j\}\|_{l(\mathbb{Z})} := \|\{y_j\}\|_{l_1(\mathbb{Z})},$$
$$\|\{y_j\}\|_{l_\infty(\mathbb{Z})} = \sup_{j\in\mathbb{Z}}\{|y_j|\}.$$

Received by the editors January 21, 1997 and, in revised form, October 13, 1998.
2000 *Mathematics Subject Classification.* Primary 41A17, 42B30; Secondary 30D15, 30D55.
Key words and phrases. Cardinal spline, entire function, Lebesgue constant.
Project 19671012 supported by both the National Natural Science Foundation and the Doctoral Programme Foundation of Institution of Higher Education of the People's Republic of China.

Schoenberg [9] proved that there is a unique element $\mathcal{L}_m y \in \mathcal{S}_{m,p}(\mathbb{R})$ interpolating the given data at integers, i.e.,

$$\mathcal{L}_m y(j) = y_j, \qquad j \in \mathbb{Z}. \tag{1.1}$$

The operator $\mathcal{L}_m : \ l_p(\mathbb{Z}) \longrightarrow \mathcal{S}_{m,p}(\mathbb{R})$ is called the cardinal spline interpolation operator of order m from $l_p(\mathbb{Z})$ to $L_p(\mathbb{R})$ and its norm

$$\|\mathcal{L}_m\|_p = \sup\{\|\mathcal{L}_m y\|_{p(\mathbb{R})} : \ \|y\|_{l_p(\mathbb{Z})} \leqslant 1\} \tag{1.2}$$

is referred to as the mth Lebesgue constant for cardinal spline interpolation. These numbers were investigated previously by many authors (see [6]–[8]).

Theorem A ([6])**.** *Let $1 < p < \infty$. Then*

$$\|\mathcal{L}_m\|_p \leqslant C_p, \tag{1.3}$$

where the constant C_p is independent of m.

Theorem B ([6])**.** *The norms of the mth order cardinal spline interpolation operators from $l_1(\mathbb{Z})$ to $L_1(\mathbb{R})$ satisfy*

$$\lim_{m\to\infty} \big(\|\mathcal{L}_m\|_1 - (4/\pi^2)\log m\big) = \big(2A/\pi\big) + 4/\pi^2\big[\log(4/\pi) + \gamma\big], \tag{1.4}$$

where γ is the Euler–Mascheroni constant and

$$A = \int_0^\pi t^{-1}\Big(\tan(\frac{t}{2}) - \frac{2}{\pi(\pi - t)}\Big)\, dt. \tag{1.5}$$

From Theorem B, we know that $\sup\limits_m\{\|\mathcal{L}_m\|_1\} = \infty$.

Let $H^1(\mathbb{Z})$ be the discrete Hardy space, consisting of those double infinite bounded sequences $y = \{y_j\} \in l_1(\mathbb{Z})$, such that $Hy = \{Hy_j\} \in l_1(\mathbb{Z})$, where

$$Hy_j = \sum_{k\neq j} \frac{y_j}{k - j}, \qquad j \in \mathbb{Z}, \tag{1.6}$$

is the discrete Hilbert transform of y. Thus $H^1(\mathbb{Z})$ is the subspace of $l_1(\mathbb{Z})$ consisting of those sequences $y = \{y_j\}$ for which the discrete Hilbert transform also belongs to $l_1(\mathbb{Z})$. Clearly

$$\|\{y_j\}\|_{H^1(\mathbb{Z})} := \|\{y_j\}\|_{l(\mathbb{Z})} + \|\{Hy_j\}\|_{l(\mathbb{Z})} \tag{1.7}$$

is a norm of $H^1(\mathbb{Z})$.

$H^1(\mathbb{Z})$ was introduced by Coifman and Weiss [3, p. 622] as an important example of the Hardy space $H^1(\mathbf{X})$, associated with a space $\mathbf{X}$ of homogeneous type, in order to extend the atomic decomposition theory for the classical Hardy spaces to a more general setting. It is well known that the Hardy space $H^1(\mathbb{R})$ is a proper closed subspace of $L_1(\mathbb{R})$, and many results in harmonic analysis and approximation theory are valid on $H^1(\mathbb{R})$ but are not correct on $L_1(\mathbb{R})$. We have found the same situation exists with respect to the Lebesgue constant of the cardinal spline interpolation operator.

Our main result is the following:

Theorem 1. *Let $\{(-1)^j y_j\} \in H^1(\mathbb{Z})$. Then for all $m \in \mathbb{N}$*

$$\|\mathcal{L}_m y\|_{L(\mathbb{R})} \leqslant \Big(1 + \frac{\pi}{2}\Big)\Big(1 + \frac{\pi}{3}\Big)\big\|\{(-1)^j y_j\}\big\|_{H^1(\mathbb{Z})}. \tag{1.8}$$

2. Interpolation operator of cardinal spline

Let $j(x)$ be the unique integer satisfying $j(x) - \frac{1}{2} \leqslant x < j(x) + \frac{1}{2}$, and let

$$\widetilde{H}y(x) = \sum{}' y_j (x-j)^{-1}, \tag{2.1}$$

where the sum $\sum'$ is taken over those $j \in \mathbb{Z}$ for which $j \neq j(x)$, and $\widetilde{H}y$ is named the mixed Hilbert transform of the sequence $y = \{y_j\}$. Following some ideas of [6], we have

Lemma 1. *Let $y \in H^1(\mathbb{Z})$. Then*

$$\|\widetilde{H}y\|_{1(\mathbb{R})} \leqslant \frac{\pi^2}{3} \|\{y_j\}\|_{H^1(\mathbb{Z})}. \tag{2.2}$$

Proof. From the definition of $j(x)$, we have $|j(x) - x| \leqslant \frac{1}{2}$, and for $j \neq j(x)$, we get

$$|j(x) - j| \leqslant |j(x) - x| + |x - j| \leqslant \frac{1}{2} + |x - j|;$$

therefore

$$\Big|\frac{j(x) - j}{x - j}\Big| \leqslant 1 + \frac{1}{2}\frac{1}{|x - j|} \leqslant 2. \tag{2.3}$$

For $j \neq j(x)$,

$$\frac{1}{x - j} = \frac{1}{j(x) - j} + \frac{(j(x) - x)(j(x) - j)}{x - j}(j(x) - j)^{-2}.$$

Hence

$$\Big|\sum{}' \frac{y_j}{x - j}\Big| \leqslant \Big|\sum{}' \frac{y_j}{j(x) - j}\Big| + \sum{}' \frac{|y_j|}{|j(x) - j|^2}, \tag{2.4}$$

from which we obtain

$$\begin{aligned}
\Big\|\sum{}' \frac{y_j}{x - j}\Big\|_{1(\mathbb{R})} &\leqslant \int_{\mathbb{R}} \Big(\Big|\sum{}' \frac{y_j}{j(x) - j}\Big| + \sum{}' \frac{|y_j|}{|j(x) - j|^2}\Big)\, dx \\
&= \sum_{k \in \mathbb{Z}} \int_{k - \frac{1}{2}}^{k + \frac{1}{2}} \Big(\Big|\sum_{j \neq k} \frac{y_j}{k - j}\Big| + \sum_{j \neq k} \frac{|y_j|}{|k - j|^2}\Big)\, dx \\
&= \sum_{k \in \mathbb{Z}} \Big|\sum_{j \neq k} \frac{y_j}{k - j}\Big| + \sum_{k \in \mathbb{Z}} \sum_{j \neq k} \frac{|y_j|}{|k - j|^2} \\
&= \sum_{k \in \mathbb{Z}} \Big|\sum_{j \neq k} \frac{y_j}{k - j}\Big| + \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{k^2} \sum_{j \in \mathbb{Z}} |y_j| \\
&\leqslant \|\{Hy_j\}\|_{l(\mathbb{Z})} + \frac{1}{3}\pi^2 \|\{y_j\}\|_{l(\mathbb{Z})} \\
&\leqslant \frac{1}{3}\pi^2 \|\{y_j\}\|_{H^1(\mathbb{Z})},
\end{aligned} \tag{2.5}$$

which completes the proof of Lemma 1. □

Let $W_\sigma y(x) = \sum_{k \in \mathbb{Z}} y_k \operatorname{sinc} \sigma(x - k\pi/\sigma)$, and let

$$\|W_\sigma\|_1 = \sup\big\{\|W_\sigma y(x)\|_{1(\mathbb{R})} : \ \big\|\{(-1)^j y_j\}\big\|_{H^1(\mathbb{Z})} \leqslant 1\big\},$$

where $\operatorname{sinc} x := x^{-1} \sin x$ for $x \neq 0$ and 1 for $x = 0$, W_σ is the well-known Whittaker operator and $W_\sigma y$ is the Whittaker cardinal series. From Lemma 1, we have

Theorem 2. *Let $\sigma > 0$. Then*

$$\|W_\sigma\|_1 \leqslant \Big(1 + \frac{1}{3}\pi\Big)\Big(\frac{\pi}{\sigma}\Big). \tag{2.6}$$

Proof. We first consider the case $\sigma = \pi$:

$$\begin{aligned}
|W_\pi y(x)| &= \Big|\sum_{j\in\mathbb{Z}} y_j \operatorname{sinc} \pi(x-j)\Big| \\
&\leqslant \Big|\frac{\sin \pi x}{\pi} \sum_{j\neq j(x)} (-1)^j \frac{y_j}{x-j}\Big| + |y_{j(x)} \operatorname{sinc} \pi(x - j(x))| \\
&\leqslant \frac{1}{\pi}\Big|\sum_{j\neq j(x)} (-1)^j \frac{y_j}{x-j}\Big| + |y_{j(x)}|.
\end{aligned}$$

Therefore, it follows from Lemma 1 that we have

$$\begin{aligned}
\|W_\pi y(x)\|_{L(\mathbb{R})} &\leqslant \frac{\pi}{3}\|\{(-1)^j y_j\}\|_{H^1(\mathbb{Z})} + \|\{y_j\}\|_{l(\mathbb{Z})} \\
&\leqslant \Big(1 + \frac{\pi}{3}\Big)\|\{(-1)^j y_j\}\|_{H^1(\mathbb{Z})}.
\end{aligned} \tag{2.7}$$

By changing scale, we obtain from (2.7) that

$$\|W_\sigma\|_1 \leqslant \Big(1 + \frac{\pi}{3}\Big)\Big(\frac{\pi}{\sigma}\Big).$$

□

Denote by $L_p^m(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, $m \in \mathbb{N}$, the subspace of f in $L_p(\mathbb{R})$ for which the $(m-1)$th derivative of f exists and is locally absolutely continuous on $\mathbb{R}$, and for which $\|f^{(m)}\|_{p(\mathbb{R})}$ is finite. By [4], if $f \in L_p^m(\mathbb{R})$, then

$$\|\{f(j)\}\|_{p(\mathbb{Z})} \leqslant \|f\|_{p(\mathbb{R})} + \|f'\|_{p(\mathbb{R})} < \infty;$$

therefore, it follows from Schoenberg [9] that for every $f \in L_p^m(\mathbb{R})$, there is a unique $\mathcal{L}_m f \in \mathcal{S}_{m,p}(\mathbb{R})$, such that $\mathcal{L}_m f(j) = f(j)$ for all $j \in \mathbb{Z}$. Moreover, we have

Lemma 2 ([5])**.** *Let $f \in L_1^m(\mathbb{R})$, $m \in \mathbb{N}$, and let $\mathcal{L}_m f$ be the unique cardinal spline of degree $m-1$ interpolating to $\{f(j)\}_{j\in\mathbb{Z}}$ at the integers. Then*

$$\|f - \mathcal{L}_m f\|_{1(\mathbb{R})} \leqslant \frac{\mathcal{K}_m}{\pi^m}\|f^{(m)}\|_{1(\mathbb{R})}, \tag{2.8}$$

where $\mathcal{K}_m$ is the Favard constant,

$$\mathcal{K}_m := \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^{k(m+1)}}{(2k+1)^{m+1}}, \tag{2.9}$$

and

$$1 = \mathcal{K}_0 < \mathcal{K}_2 < \cdots < \frac{4}{\pi} < \cdots < \mathcal{K}_3 < \mathcal{K}_1 = \frac{\pi}{2}.$$

Remark 1. de Boor and Schoenberg [2] proved that equation (2.8) is also valid for m even and $p = \infty$.

Let $\mathcal{E}_\sigma(\mathbb{R})$, $\sigma > 0$, be the restriction on $\mathbb{R}$ of entire functions of exponential type σ, and let

$$B_{\sigma,p} = \mathcal{E}_\sigma(\mathbb{R}) \cap L_p(\mathbb{R}), \quad 1 \leqslant p \leqslant \infty, \qquad B_\sigma := B_{\sigma,\infty}.$$

It is well known that $B_{\sigma,p} \subseteq B_{\sigma,q}$, $1 \leqslant p < q \leqslant \infty$.

Lemma 3 ([1, p. 211] Inequality of Bernstein's type)**.** *Let $f \in B_{\sigma,p}$, $1 \leqslant p \leqslant \infty$, $\sigma > 0$. Then*

$$\|f'\|_{p(\mathbb{R})} \leqslant \sigma \|f\|_{p(\mathbb{R})}.$$

Lemma 4 ([10])**.** *Let $y = \{y_j\} \in l_2$. Then there is a unique $f \in B_{\pi,2}$, interpolating the given data $y = \{y_j\}_{j\in\mathbb{Z}}$ at the integers, and f is represented by*

$$f(x) = \sum_{j\in\mathbb{Z}} y_j \operatorname{sinc} \pi(x-j), \quad \textit{for all } x \in \mathbb{R},$$

and the series $\sum_{j\in\mathbb{Z}} y_j \operatorname{sinc} \pi(x-j)$ converges uniformly on $\mathbb{R}$.

Proof of Theorem 1. Let $\{(-1)^j y_j\} \in H^1(\mathbb{Z})$. Then $\{y_j\} \in l_2(\mathbb{Z})$. By Lemma 4, there exists a function $f \in B_{\pi,2}$ such that $f(j) = y_j$ for all $j \in \mathbb{Z}$, hence $f \in B_\pi$. It follows from Theorem 2 that $f \in L_1(\mathbb{R})$; therefore $f \in B_{\pi,1}$. Using Lemma 2 and Bernstein's inequality we get

$$\begin{aligned} \|f - \mathcal{L}_m f\|_{1(\mathbb{R})} &\leqslant \frac{\mathcal{K}_m}{\pi^m} \|f^{(m)}\|_{1(\mathbb{R})} \\ &\leqslant \mathcal{K}_m \|f\|_{1(\mathbb{R})}, \end{aligned} \tag{2.10}$$

which together with (2.9) and Theorem 2 gives

$$\begin{aligned} \|\mathcal{L}_m y\|_{1(\mathbb{R})} = \|\mathcal{L}_m f\|_{1(\mathbb{R})} &\leqslant (1 + \mathcal{K}_m)\|f\|_{1(\mathbb{R})} \\ &\leqslant \left(1 + \frac{\pi}{2}\right)\left(1 + \frac{\pi}{3}\right) \|\{(-1)^j y_j\}\|_{H^1(\mathbb{Z})}, \end{aligned}$$

which completes the proof of Theorem 1. □

Acknowledgment

The author would like to thank the referee for his valuable suggestions and comments.

References

1. R. P. Boas, Jr., Entire Functions, Academic Press, New York, 1954. MR **16:**914f
2. C. de Boor and I. J. Schoenberg, Cardinal interpolation and spline functions VIII, Lecture Notes in Mathematics, Vol. 501, Springer, Berlin, 1976, 1-79. MR **58:**12097c
3. R. R. Coifman and G. Weiss, Extension of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83**(1977), 569–645. MR **56:**6264
4. Fang Gensun, Whittaker–Kotelnikov–Shannon sampling theorem and Aliasing error, *J. Approx. Theory* **85**(1996), 115–131. MR **97b:**41002
5. G. G. Magril–Il'yaev, Average dimension, widths, and optimal recovery of Sobolev classes of functions on the line, *Math. USSR Sbornik* **74**(1993), 381–403. MR **92k:**41034
6. M. J. Marsden, F. B. Richards, and S. D. Riemenschneider, Cardinal spline interpolation operators on l^p data, *Indiana Univ. Math. J.* **24**(1975), 677–689. MR **52:**3807
7. F. Richards, The Lebesgue constants for Cardinal Spline Interpolation, *J. Approx. Theory* **14**(1975), 83–92. MR **52:**6254
8. F. Richards, Uniform spline interpolation in L_2, *Illinois J. Math.* **18** (1974), 516-521. MR **50:**10620
9. I. J. Schoenberg, Cardinal Spline Interpolation, CBMS, Vol.12, SIAM, Philadelphia, 1973. MR **54:**8095
10. J. M. Whittaker, Interpolatory Function Theory, Cambridge University Press, 1935.

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