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CRITERIA FOR CONVEXITY IN BANACH SPACES

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Abstract. In this paper two convexity criteria are proven. The first one characterizes compact convex sets in a locally convex space and extends a previous result by G. Aumann, while the second one characterizes closed bounded convex sets with the Radon-Nikodým property in a Banach space.

Introduction

In the present paper the following two theorems are proven:

Theorem 1. *Let K be a closed subset of a locally convex space X such that $L = \overline{conv}K$ is compact in X. The following are equivalent:*

(i) *$K = L$, that is, K is convex.*

(ii) *For every closed hyperplane H of X which intersects K, $H \cap K$ is either contractible or has the fixed point property.*

Theorem 2. *Let K be a weakly closed bounded subset of a Banach space X such that $L = \overline{conv}K$ has the Radon-Nikodým property. The following are equivalent:*

(i) *$K = L$, that is, K is convex.*

(ii) *For every closed hyperplane H of X which intersects K, $H \cap K$ is either contractible or has the fixed point property in the narrow sense.*

(In the above theorems it is assumed that $\dim X \geq 2$.)

A set A is assumed to have *the fixed point property in the narrow sense* if every continuous compact map of A into itself has a fixed point. It is known that every closed convex subset of a Banach space has that property (Schauder's Theorem, cf. [Du1]) and every compact convex subset of a locally convex space has the fixed point property in the usual sense (Tychonoff's Theorem, cf. [Du1]). Also, as a direct consequence of the convexity we have that every convex set in a topological vector space is contractible. A closed bounded convex subset A of a Banach space X is assumed to have *the Radon-Nikodým property (RNP)* if every subset of A has slices of arbitrarily small diameter (for a detailed study of this notion see [B]).

For the case $X = \mathbb{R}^n$ similar characterizations have been proven by G. Aumann ([A]) with methods from algebraic topology. Later I. Fàry ([F]) and A. Kosinski ([K]) gave other proofs for the same result.

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The approach of this paper is based on the notion of the n-extreme points for Theorem 1 and n-denting points of a convex set K ($Ex_n(K)$, resp. $D_n(K)$) for Theorem 2. The precise definition of them is given in Section 1. These are subsets of K strongly related with its extreme points and as is proven in the same section, the sets $\bigcup_{n=0}^{\infty} Ex_n(K)$ (for compact K), $\bigcup_{n=0}^{\infty} D_n(K)$ (for K with the RNP) are weakly dense in the corresponding set K. In Section 2 the n-cycles are defined and a sufficient condition is considered for a set K to have a non contractible and without the fixed point property hyperplane section. In Section 3 it is shown that under some assumptions K satisfies the previous condition. Finally in Section 4 the proof of Theorems 1 and 2 is provided, through two more general theorems (Theorems 17 and 18) and some remarks are given.

Notation. The class of all closed subspaces of a locally convex space X which have codimension m, $m \in \mathbb{N}$, is denoted by $\Sigma^m(X)$. The class of all affine closed subspaces of X which have codimension m, $m \in \mathbb{N}$, is denoted by $A^m(X)$. We also set $\Sigma(X) = \bigcup_{m=0}^{\infty} \Sigma^m(X)$ and $A(X) = \bigcup_{m=0}^{\infty} A^m(X)$. The members of $A(X)$ will in general be called *flats.* If A is a subset of a Banach space, then $\mathcal{P}(A)$ denotes the set of all points of continuity of the identity map $Id_A : (A,\ weak) \rightarrow (A,\ norm)$.

Some standard notation is also used. If $A \subset X$, then $relint A$ denotes the relative interior of A, $relbd(A)$ the relative boundary of A and $\dim A$ the dimension of A, that is, the dimension of its affine hull.

SECTION 1

Let X be a locally convex space, $L \subset X$ be a closed convex set and suppose that $x \in L$. Then x is called an *n-extreme* point of L ($n \in \mathbb{N}$) if x is in the relative interior of no $(n+1)$-dimensional convex subset of L. The set of all n-extreme points of L is denoted by $Ex_n(L)$. We also define $E(L) = \bigcup_{n=0}^{\infty} Ex_n(L)$.

Evidently if $n = 0$, we get the usual notion of an extreme point of L. Also if $m < n$, then $Ex_m(L) \subset Ex_n(L)$. As can easily be proven, the basic property of the n-extreme points is the following: A point $x \in Ex_n(L)$ if and only if there exists a unique face A_x of L which contains x in its relative interior and with $\dim A_x \leq n$. (As usually denoted a *face* of L is a closed convex set $A \subset L$ such that each segment $[x_1, x_2] \subset L$ with $A \cap relint[x_1, x_2] \neq \emptyset$ is contained in A.) The reader can refer to [S] for a more extensive study of the notion of n-extreme points in finite dimensional spaces.

At this point let us give some examples for the n-extreme points:

(a) If $X = \ell^p$, $1 < p < \infty$, $L = B_{\ell^p}$ (the unit ball of ℓ^p), then it can be shown that $Ex_0(L) = Ex_n(L) = S_{\ell^p}$ (the unit sphere of ℓ^p), for every $n \in \mathbb{N}$.

(b) If $X = \ell^1$, $L = B_{\ell^1}$, then we have that:

$$Ex_{n-1}(L) = \left\{ \lambda_{m_1} e_{m_1} + ... + \lambda_{m_n} e_{m_n} :\ m_1 < ... < m_n,\ \sum_{i=1}^{n} |\lambda_{m_i}| = 1 \right\}$$

for every $n \geq 1$, where $\{e_1, e_2, ...\}$ is the usual basis of ℓ^1.

(c) If $X = c_0$ and $L = B_{c_0}$, then $Ex_n(L) = \emptyset$ for every $n \geq 0$.

Lemma 3. *Let L be a closed convex subset of X and suppose that a flat $F \in A^m(X)$ intersects L. Then $F \cap L$ is a closed convex subset of X such that $Ex_0(F \cap L) \subset Ex_m(L)$.*

Proof. If $Ex_0(F\cap L)=\emptyset$, then the lemma trivially holds. So let us assume that $Ex_0(F\cap L)\neq\emptyset$ and $z\in Ex_0(F\cap L)$.

Suppose that $z\notin Ex_m(L)$. Then there exists an $(m+1)$-dimensional convex set K with $z\in relintK$ such that $K\subset L$. Since F is a flat of codimension m and $z\in F$ is at the same time in the relative interior of an $(m+1)$-dimensional convex set, there exists a segment $[x_1,x_2]\subset F\cap K\subset F\cap L$, such that z is the midpoint of $[x_1,x_2]$. Hence $z\notin Ex_0(F\cap L)$, which is a contradiction. ☐

Proposition 4. *If L is a compact convex subset of a locally convex space X, then $L=\overline{E(L)}$.*

Proof. If L is finite dimensional, then evidently $L=E(L)=Ex_m(L)$ where $m=\dim L$. If L is infinite dimensional, we can assume that L is weakly compact and so it is enough to prove that $E(L)$ is weakly dense in L.

Let V be a weakly open basic neighborhood of a point $x_0\in L$. Since $E(L)=\bigcup_{n=0}^{\infty}Ex_n(L)$, it suffices to prove that there exists an $m\in\mathbb{N}$ which depends on V such that $V\cap Ex_m(L)\neq\emptyset$. Suppose that V is determined by the functionals $f_1,\ldots,f_k\in X^*$. We set $F=x_0+\bigcap_{i=1}^{k}Kerf_i$. Then $F\in A^m(X)$ for some $m\leq k$, $x_0\in F$ and $F\subset V$. Evidently $F\cap L$ is a non empty compact convex subset of X and so $Ex_0(F\cap L)\neq\emptyset$ by Krein-Milman's Theorem. According to the previous lemma $Ex_0(F\cap L)\subset Ex_m(L)$, hence $\emptyset\neq Ex_0(F\cap L)\subset F\cap Ex_m(L)\subset V\cap Ex_m(L)$. ☐

Remark 1. Using the same method as in the previous proposition, one can prove that if L is a closed convex subset of a locally convex space such that L has the Krein-Milman Property (that is, each closed convex subset of L has an extreme point), then $L=\overline{E(L)}^{w}$.

Let us suppose for the rest of Section 1 that X is a Banach space. Let L be a closed bounded convex subset of X. A point $x\in L$ is an *n-denting* point of L ($n\in\mathbb{N}$) if: (a) x is an n-extreme point of L and (b) x is a point of continuity of the identity map $id_L:(L,weak)\to(L,norm)$, that is, $x\in Ex_n(L)\cap\mathcal{P}(L)$. The set of all n-denting points of L is denoted by $D_n(L)$. We also define $D(L)=\bigcup_{n=0}^{\infty}D_n(L)$.

Evidently if $n=0$, we get the usual notion of a denting point of L. Also if $m<n$, then $D_m(L)\subset D_n(L)$. Since every $x\in D_n(L)$ is an n-extreme point of L, there exists a unique face A_x of L which contains x in its relative interior and with $\dim A_x\leq n$. The converse does not hold in general, but if L is finite-dimensional, then evidently $Ex_n(L)=D_n(L)$ for every $n\in\mathbb{N}$, since the weak and norm topologies coincide on L. The examples (a) and (b) of n-extreme points, given in the beginning of this section, are also examples for n-denting points.

The following result was suggested by Professor S. A. Argyros.

Proposition 5. *Let L be a bounded convex subset of a Banach space X and suppose that V is a weakly open basic neighborhood of a point $x_0\in L$. Then $\mathcal{P}\left(\overline{V}\cap L\right)\subset\mathcal{P}(L)$.*

Proof. Evidently if $\mathcal{P}\left(\overline{V}\cap L\right)=\emptyset$, the theorem trivially holds. So let us suppose that $z_0\in\mathcal{P}\left(\overline{V}\cap L\right)$. Then for every $\varepsilon>0$ there exists a weakly open basic neighborhood of z_0 which is denoted by W_ε such that $diam\left(W_\varepsilon\cap\overline{V}\cap L\right)<\varepsilon$. If $z_0\in V$, then obviously $W_\varepsilon\cap V$ is a weakly open neighborhood of z_0 such that $diam(W_\varepsilon\cap V\cap L)<\varepsilon$. Since ε is arbitrarily chosen, $z_0\in\mathcal{P}(L)$.

If $z_0 \in \overline{V} \backslash V$, then without loss of generality we may suppose the following:

(1) $x_0 = 0$ and $V = \{x \in X : |f_i(x)| < 1 \text{ for } i = 1, ..., k\}$ where $f_1, ..., f_k \in X^*$ and so $|f_i(z_0)| \le 1$ for $i = 1, ..., k$.

(2) $0 \notin W_\varepsilon$ and $W_\varepsilon = \{x \in X : |g_j(x - z_0)| < 1 \text{ for } j = 1, ..., m\}$ where $g_1, ..., g_m \in X^*$. Hence $|g_j(z_0)| \ge 1$, for some $j \in \{1, ..., m\}$.

(3) $0 < \varepsilon < \frac{M}{C}$ where $M = \sup\{\|x\| : x \in L\}$ and $C = \max\{|g_j(z_0)| : 1 \le j \le m\}$. According to (2) it is clear that $C \ge 1$ and $M > 0$.

Let us define:

$V' = \left\{x \in X : |f_i(x)| < 1 + \frac{\varepsilon}{2M} \text{ for } i = 1, ..., k\right\}$ and

$W'_\varepsilon = \left\{x \in X : |g_j(x - z_0)| < 1 - \frac{\varepsilon C}{M} \text{ for } j = 1, ..., m\right\}$.

Claim. (a) *Let* $r \in (0,1)$ *such that* $\frac{1}{1+\frac{\varepsilon}{M}} < r < \frac{1}{1+\frac{\varepsilon}{2M}}$. *Then* $r\left(V' \cap W'_\varepsilon \cap L\right) \subset V \cap W_\varepsilon \cap L$.

(b) *diam* $\left(V' \cap W'_\varepsilon \cap L\right) < 4\varepsilon$.

Proof of the Claim. (a) Let $x \in V' \cap W'_\varepsilon \cap L$. Then

(1) For $i = 1, ..., k$ $|f_i(rx)| = r|f_i(x)| < r\left(1 + \frac{\varepsilon}{2M}\right) < 1$. Hence $rx \in V$.

(2) For $j = 1, ..., m$,

$$\begin{aligned} |g_j(rx - z_0)| &= |g_j(rx - rz_0 + rz_0 - z_0)| \le r|g_j(x - z_0)| + (1-r)|g_j(z_0)| \\ &< r\left(1 - \tfrac{\varepsilon C}{M}\right) + (1-r)C = r - r\left(\frac{\varepsilon}{M} + 1\right)C + C < r - C + C = r < 1. \end{aligned}$$

Hence for $j = 1, ..., m$ $|g_j(rx - z_0)| < 1$ and so $rx \in W_\varepsilon$.

(3) Since L is convex and $0 \in L$, then $(1-r)0 + rx = rx \in L$.

By (1), (2) and (3) $rx \in V \cap W_\varepsilon \cap L$ for every $x \in V' \cap W'_\varepsilon \cap L$.

(b) Let $x \in V' \cap W'_\varepsilon \cap L$ and $r \in (0,1)$ as in (a). Then $||rx - x|| = (1-r)||x|| < \left(1 - \frac{1}{1+\frac{\varepsilon}{M}}\right)M = \frac{\varepsilon}{1+\frac{\varepsilon}{M}}$ and $||rx - z_0|| < \varepsilon$ since $z_0, rx \in \left(\overline{V} \cap W_\varepsilon \cap L\right)$ (by (a)) and *diam* $\left(\overline{V} \cap W_\varepsilon \cap L\right) < \varepsilon$. Therefore $||x - z_0|| \le ||rx - x|| + ||rx - z_0|| < \varepsilon + \frac{\varepsilon}{1+\frac{\varepsilon}{M}}$.

Hence $\sup\left\{||x - y|| : x, y \in V' \cap W'_\varepsilon \cap L\right\} < 4\varepsilon$. The proof of the claim is complete.

Since $V' \cap W'_\varepsilon$ is evidently a weakly open neighborhood of z_0 and ε is arbitrarily chosen, we conclude by (b) of the claim that $z_0 \in \mathcal{P}(L)$ and thus the proof of Proposition 5 is complete. □

Lemma 6. *Let* L *be a closed bounded convex subset of a Banach space* X *and suppose that* V *is a weakly open basic neighborhood of a point* $x_0 \in L$. *Then* $\overline{V} \cap L$ *is a closed bounded convex subset of* X *with* $Ex_0\left(\overline{V} \cap L\right) \subset E(L)$.

Proof. It is evident that the set $\overline{V} \cap L$ is a closed bounded convex subset of X. If $Ex_0\left(\overline{V} \cap L\right) = \emptyset$, then the lemma trivially holds. So let us suppose that $z \in Ex_0\left(\overline{V} \cap L\right)$. Following the proof of Proposition 4, there exists an $F \in A^m(X)$ for some $m \in \mathbb{N}$ such that $z \in F$ and $F \subset \overline{V}$. Obviously $z \in Ex_0(F \cap L)$ and by Lemma 3, $z \in Ex_m(L)$. Therefore $z \in E(L)$. □

Proposition 7. *Let* L *be a closed bounded convex subset of a Banach space* X *and suppose that* L *has the RNP. Then* $L = \overline{D(L)}^w$.

Proof. Since L is a norm closed convex subset of X, L is weakly closed as well. So it is enough to prove that $D(L)$ is weakly dense in L. To this end, let us assume that V is a weakly open basic neighborhood of a point $x_0 \in L$. It is clear that there exists a weakly open basic neighborhood V' of the same point such that $\overline{V'} \subset V$. Evidently $\overline{V'} \cap L$ is a closed convex subset of L and since L has the RNP, $D_0\left(\overline{V'} \cap L\right) \neq \emptyset$ by the Phelps-Bourgain Theorem (cf. [B]). By Proposition 5 and Lemma 6 we have that $D_0\left(\overline{V'} \cap L\right) \subset E(L) \cap \mathcal{P}(L) = D(L)$. Hence $\emptyset \neq \overline{V'} \cap D(L) \subset V \cap D(L)$. ☐

SECTION 2

Let S, F be closed subspaces of a locally convex space X where F is finite dimensional and $S \cap F = \{0\}$. We write $S \oplus F$ for the linear span of $S \cup F$. It is obvious that every $x \in S \oplus F$ can be uniquely written as $x = s + f$, where $s \in S$ and $f \in F$. Hence we can define the projection $P_F : S \oplus F \to F$, where $P_F(x) = f$, if $x = s + f$. In the trivial case where $S = \{0\}$, P_F is the identity map on F. Since X is a locally convex space and F is finite dimensional, it can be proven that $S \oplus F$ is a closed subspace of X and the projection P_F is continuous.

Notation. The relative boundary of an $(n+1)$-dimensional compact convex set will be called an *n-cycle*. We denote by $\mathcal{K}_0^n$ the class of all n-dimensional compact convex subsets of X which contain 0 in their relative interior. The relative boundary of a set $L \in \mathcal{K}_0^{n+1}$ is called an *n-cycle around* 0.

Lemma 8. *Let K be a subset of a locally space X and $H \in \Sigma^1(X)$ with the following properties:*

(i) *$H = F \oplus F_n$, where $F \in \Sigma^{n+1}(X)$, F_n is an n-dimensional subspace of X and $n \geq 1$.*

(ii) *$F \cap K = \emptyset$.*

(iii) *$F_n \cap K$ contains an $(n-1)$-cycle C around* 0.

Then C is a retract of $H \cap K$.

Proof. For every $x \in F_n \backslash \{0\}$ we define $P_C(x) = C \cap \{\lambda x : \lambda > 0\}$. Since C is an $(n-1)$-cycle, P_C is well defined and continuous. Let $P_{F_n} : H \to F_n$ be the projection onto F_n. We note that $P_{F_n}(H \cap K) \subset F_n \backslash \{0\}$, since $P_{F_n}(x) = 0$ if and only if $x \in F$, whereas $F \cap K = \emptyset$ by assumption (ii).

So the map $S : H \cap K \to C$ defined by $S(x) = P_C(P_{F_n}(x))$, where $x \in H \cap K$, is a well defined continuous map onto C leaving every point of C fixed. That is, C is a retract of $H \cap K$. ☐

Remark 2. Suppose that a set $A \subset X$ is contractible or has the fixed point property in the usual or in the narrow sense. Then there is no n-cycle which is a retract of A. Indeed, it is easy to see that every retract of A has the above mentioned properties, whereas an n-cycle as an homeomorphic image of the unit sphere S^n of $\mathbb{R}^{n+1}$, neither is contractible by Brouwer's Theorem (cf. [Du1]) nor has the fixed point property in the usual or in the narrow sense.

SECTION 3

In the following and up to Proposition 11 let us denote by X a locally convex space X and by K a closed subset of X such that the set $L = \overline{conv} K$ is compact.

Let us define $Ex_m^*(L) = Ex_m(L) \backslash Ex_{m-1}(L)$, $m \geq 1$, and suppose that $0 \in Ex_m^*(L)$. According to the basic property of the n-extreme points (Section 1), there exists a unique face A_0 of L which contains 0 in its relative interior. Let us denote by F_m the linear span of A_0. It is obvious that $A_0 \in \mathcal{K}_0^m$ and so F_m is an

m-dimensional (and thus closed) subspace of X. Since F_m is finite dimensional, there exists a $Y \in \Sigma^m(X)$ such that $Y \oplus F_m = X$.

In the following Lemma 9 we shall need Choquet's Lemma (cf. [HHZ]), that is, in every compact convex subset L of X the slices of L consist of a neighborhood base for the extreme points of L. As is known, a slice of L is the intersection of L with an open halfspace of X. Formally, if $f \in X^*$ and $a \in \mathbb{R}$, then a slice of L can be written as $S(f, L, a) = \{y \in L : f(y) > a\}$.

Lemma 9. *Let* K, L, A_0, F_m, Y *be as above and* $0 \in Ex^*_m(L) \backslash K$, $1 \leq m < \dim X$. *Then:*

(a) $0 \in Ex_0(Y \cap L)$.

(b) *There exist an* $H \in \Sigma^1(X)$ *and an* $F \in \Sigma^{m+1}(X)$, $F \subset Y$ *such that:*

(i) $H = F \oplus F_m$

(ii) $F \cap K = \emptyset$.

Proof. (a) Obviously $Y \cap L$ is a compact convex subset of L containing 0. If $0 \notin Ex_0(Y \cap L)$, then there exists a closed segment $[x_1, x_2] \subset Y \cap L$ with midpoint at 0 and $x_1 \neq x_2$. Since A_0 is a face of L and $0 \in relint[x_1, x_2] \cap A_0$, we have that $[x_1, x_2] \subset A_0$. Hence $[x_1, x_2] \subset Y \cap L \cap A_0 \subset Y \cap L \cap F_m = \{0\}$ which is a contradiction.

(b) We notice that $\dim Y \geq 1$ since $m < \dim X$. Two cases are distinguished. Either $Y \cap K = \emptyset$ or $Y \cap K \neq \emptyset$.

If $Y \cap K = \emptyset$, then for every $F \in \Sigma^1(Y)$ we have $F \in \Sigma^{m+1}(X)$, $F \oplus F_m = H \in \Sigma^1(X)$ and $F \cap K \subset Y \cap K = \emptyset$. (If $\dim Y = 1$, then F is necessarily the trivial space $\{0\}$.)

If $Y \cap K \neq \emptyset$, then since $0 \notin K$ and K is closed, there exists an open neighborhood V of 0 in X such that $V \cap K = \emptyset$. But then $V \cap (Y \cap L)$ is an open neighborhood of 0 in the compact convex set $Y \cap L$. Since (by (a)) $0 \in Ex_0(Y \cap L)$ Choquet's lemma can be applied. Therefore there exists a slice $S = S(f, Y \cap L, a)$ such that $0 \in S$ and $S \subset V \cap (Y \cap L)$. Hence, $S \cap K = \emptyset$.

Let $G = Kerf$. Then $G \in \Sigma^1(X)$ and since $0 \in S$, $G \cap (Y \cap L) \subset S$. We note that Y is not contained in G, since otherwise $Y \cap K = (G \cap Y) \cap (L \cap K) = G \cap (Y \cap L) \cap K \subset S \cap K = \emptyset$, which is a contradiction to our assumption where $Y \cap K \neq \emptyset$.

Let $F = G \cap Y$. Since $Y \in \Sigma^m(X)$, $G \in \Sigma^1(X)$ and $Y \nsubseteq G$ we conclude that $F \in \Sigma^{m+1}(X)$. We also have that $F \cap F_m = \{0\}$, since $F \subset Y$, and so the subspace $H = F \oplus F_m$ is a well defined closed hyperplane of X. Finally $F \cap K = (G \cap Y) \cap (L \cap K) = G \cap (Y \cap L) \cap K \subset S \cap K = \emptyset$. □

Lemma 10. *Let* $0 \in Ex_m(L)$, $m \geq 1$ *and* $Ex_{m-1}(L) \subset K$. *Then* $relbdA_0 \subset K$ *and therefore* $F_m \cap K$ *contains an* $(m-1)$*-cycle around* 0.

Proof. Since $A_0 \in \mathcal{K}^m_0$, $relbdA_0$ is an $(m-1)$-cycle around 0 and $relbdA_0 = Ex_{m-1}(A_0)$. On the other hand A_0 is a face of L. Hence every face of A_0 is a face of L as well. By the basic property of m-extreme points (as it is stated in Section 1), this means that $Ex_{m-1}(A_0) \subset Ex_{m-1}(L)$. Since $Ex_{m-1}(L) \subset K$, $relbdA_0 \subset K$ as well. □

Proposition 11. *Let* K *be a closed subset of a locally convex space* X *with the following properties:*

(i) $L = \overline{conv}K$ *is a compact subset of* X.

(ii) $Ex_{m-1}(L) \subset K$, *for some* $m \geq 1$.
(iii) $0 \in Ex_m(L) \backslash K$.

Then there exist a hyperplane $H \in \Sigma^1(X)$ *and an* n*-cycle* C *around* 0 *(where* $n = m-1$ *if* $m < \dim X$ *or* $n = m-2$ *if* $\dim X = m$*), such that* C *is a retract of* $H \cap K$.

Proof. Obviously $0 \in Ex_m^*(L) \backslash K$. Two cases are distinguished:

(1) $1 \leq m < \dim X$.

By Lemmas 9 and 10 there exist an $H \in \Sigma^1(X)$ and an $F \in \Sigma^{m+1}(X)$ such that: (i) $H = F \oplus F_m$ and F_m is an m-dimensional subspace of X; (ii) $F \cap K = \emptyset$; (iii) $F_m \cap K$ contains a $(m-1)$-cycle C around 0. By Lemma 8, C is a retract of $H \cap K$.

(2) $m = \dim X \geq 2$.

Then $F_m = X$, X is isomorphic to $\mathbb{R}^m$ and $A_0 = L$. Let us denote by BdL the boundary of L. By Lemma 10, $BdL \subset K$. For every $H \in \Sigma^1(X)$ the following trivially hold: (i) $H = \{0\} \oplus H$ and H is an $(m-1)$-dimensional subspace of X; (ii) $\{0\} \cap K = \emptyset$; (iii) $H \cap K$ contains an $(m-2)$-cycle C around 0, where $C = BdL \cap H$. By Lemma 8 for each $H \in \Sigma^1(X)$ there exists an $(m-2)$-cycle around 0 which is a retract of $H \cap K$. □

Hereafter let us denote by X a Banach space.

Lemma 12. *Let* L *be a convex subset of a Banach space* X *and suppose that* $z \in \mathcal{P}(L)$. *If* $z = rz_1 + (1-r)z_2$, *where* $r \in (0,1)$, *and* $z_1, z_2 \in L$, *then* $z_1, z_2 \in \mathcal{P}(L)$ *as well.*

Proof. We prove that $z_1 \in \mathcal{P}(L)$ (the proof of z_2 is similar). Since $z \in \mathcal{P}(L)$, for every $\varepsilon > 0$ there exists a weakly open neighborhood V_ε of z such that $diam\,(V_\varepsilon \cap L) < \varepsilon$. We define $V_\varepsilon^1 = \frac{1}{r}(V_\varepsilon - (1-r)z_2)$. Evidently, V_ε^1 is a weakly open neighborhood of z_1. It is easy to see that $diam\left(V_\varepsilon^1 \cap L\right) \leq \frac{1}{r}\varepsilon$. As ε is arbitrarily chosen, it follows that $z_1 \in \mathcal{P}(L)$. □

The lemma corresponding to Choquet's Lemma in the case of 0-denting points, due to Troyanski, Lin and Lin (cf. [HHZ], Chapter 5), is the following:

Lemma 13. *Let* L *be a closed bounded convex subset of a Banach space* X *and suppose that* $z \in D_0(L)$. *Then the slices of* L *consist a neighborhood base for the norm topology at* z. □

Let us suppose in the sequel that K is a closed bounded subset of X and $L = \overline{conv}K$. Let us also define $D_m^*(L) = D_m(L) \backslash D_{m-1}(L)$ for $m \geq 1$ and suppose that $0 \in D_m^*(L)$. Hence $0 \in Ex_m^*(L)$ as well and so we can define A_0, F_m and Y just as we have defined them in the beginning of Section 3.

The proof of the following lemma is similar to that of Lemma 9, except that $Y \cap L$ is a closed bounded subset of a Banach space X, $0 \in D_0(Y \cap L)$ and we apply Lemma 13 instead of Choquet's Lemma.

Lemma 14. *Let* K, L, A_0, F_m, Y *be as above and* $0 \in D_m^*(L) \backslash K$, $1 \leq m < \dim X$. *Then:*

(a) $0 \in D_0(Y \cap L)$.
(b) *There exist an* $H \in \Sigma^1(X)$ *and an* $F \in \Sigma^{m+1}(X)$, $F \subset Y$ *such that:*
(i) $H = F \oplus F_m$.
(ii) $F \cap K = \emptyset$. □

Lemma 15. *Let* $0 \in D_m(L)$, $m \geq 1$, *and* $D_{m-1}(L) \subset K$. *Then* $relbdA_0 \subset K$ *and therefore* $F_m \cap K$ *contains an* $(m-1)$*-cycle around* 0.

Proof. Using the arguments of Lemma 10, it can be shown that

$$relbdA_0 = Ex_{m-1}(A_0) \subset Ex_{m-1}(L).$$

Since $0 \in relintA_0$, for every point $z_1 \in relbdA_0$, there exists another point $z_2 \in relbdA_0$ such that $0 = rz_1 + (1-r)z_2$ for some $r \in (0,1)$. By Lemma 12, since $0 \in \mathcal{P}(L)$ we have that $z_1, z_2 \in \mathcal{P}(L)$. Hence $relbdA_0 \subset Ex_{m-1}(L) \cap \mathcal{P}(L) = D_{m-1}(L)$. Since $D_{m-1}(L) \subset K$, $relbdA_0 \subset K$ as well. $\square$

Proposition 16. *Let* K *be a closed bounded subset of a Banach space* X *and* $L = \overline{conv}K$, *with the following properties:*

(i) $D_{m-1}(L) \subset K$, *for some* $m \geq 1$.

(ii) $0 \in D_m(L) \backslash K$.

Then there exist a hyperplane $H \in \Sigma^1(X)$ *and an* n*-cycle* C *around* 0 *(where* $n = m-1$ *if* $m < \dim X$ *or* $n = m-2$ *if* $\dim X = m$*), such that* C *is a retract of* $H \cap K$.

Proof. Similar to that of Proposition 11 except that now we use Lemmas 14 and 15 instead of Lemmas 9 and 10.

SECTION 4

(a) *Proof of Theorem 1.* (i) ⇒ (ii) If K is a compact convex subset of X and $H \in \Sigma^1(X)$ such that $H \cap K \neq \emptyset$, then evidently $H \cap K$ is a non empty compact convex subset of X and therefore it is contractible and has the fixed point property, as is noted in the Introduction.

(ii) ⇒ (i) According to Remark 2 it is enough to prove the following:

Theorem 17. *Let* K *be a closed subset of a locally convex space* X *such that* $L = \overline{conv}K$ *is compact in* X *and suppose that for every closed hyperplane* H *of* X *which intersects* K, *there is no* n*-cycle,* $n \in \mathbb{N}$, *which is a retract of* $H \cap K$. *Then* $E(L) \subset K$ *and* K *is convex.*

Proof. We will prove inductively that $Ex_m(L) \subset K$, for every $m \in \mathbb{N}$. For $m = 0$ it is a consequence of Choquet's Lemma. Let us suppose that for some $m \geq 1$, $Ex_{m-1}(L) \subset K$ and $Ex_m(L) \nsubseteq K$. Then there exists a point $x_0 \in Ex_m(L) \backslash K$ which without loss of generality is the point 0. But then by Proposition 11 there exist an $H \in \Sigma^1(X)$ and an n-cycle C such that C is a retract of $H \cap K$, a contradiction. Hence $Ex_m(L) \subset K$ for every $m \in \mathbb{N}$ and so $E(L) \subset K$. Since by Proposition 4 $L = \overline{E(L)}$, we have that $K = L$ and so K is convex. $\square$

The proof of Theorem 1 is complete.

(b) *Proof of Theorem 2.* (i) ⇒ (ii) It is similar to that of Theorem 1, except that we use Schauder's fixed point theorem in place of Tychonoff's.

(ii) ⇒ (i) Like in Theorem 1 it is enough to prove the following:

Theorem 18. *Let* K *be a closed bounded subset of a Banach space* X *such that* $L = \overline{conv}K$ *has the RNP and suppose that for every closed hyperplane* H *of* X *which intersects* K, *there is no* n*-cycle which is a retract of* $H \cap K$. *Then* $D(L) \subset K$ *and* K *is weakly dense in* L.

Proof. Similar to that of the Theorem 17 except that we use Lemma 13, Proposition 16 and Proposition 7 instead of Choquet's Lemma, Proposition 11 and Proposition 4, respectively. ☐

The proof of Theorem 2 is complete. ☐

Remark 3. In Theorem 18 the conclusion that K is weakly dense in L cannot be improved. For example let $X = \ell^2$ and $K = S_{\ell^2}$; then $L = \overline{conv} K = B_{\ell^2}$ and as is known B_{ℓ^2} has the RNP. If H is a closed hyperplane of ℓ^2 that intersects S_{ℓ^2}, then $H \cap S_{\ell^2}$ is contractible and has the fixed point property in the narrow sense. Indeed $H \cap S_{\ell^2}$ is a single point or a retract of $H \cap B_{\ell^2}$, as follows by the related theorem of Dugundji ([Du2]). Of course S_{ℓ^2} is not convex but $\overline{S_{\ell^2}}^{w} = B_{\ell^2}$.

Remark 4. Let us denote by (X, τ) a Banach space X equipped with a topology τ where τ is the norm or the weak or the weak* topology (if X is a dual space). Then (X, τ) is a locally convex space. Various theorems (such as Krein's theorem if τ is the weak topology) establish that if K is a compact subset of (X, τ), then $L = \overline{conv} K$ (where the closure is taken with respect to the topology τ) is compact as well. So, as a direct consequence of Theorem 1, we have the following:

Theorem 19. *Let K be a compact subset of (X, τ). Then K is convex, if and only if for every closed hyperplane H of X which intersects K, $H \cap K$ is either contractible or has the fixed point property.*

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