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ON NON-ORIENTABLE SURFACES IN 4-SPACE WHICH ARE PROJECTED WITH AT MOST ONE TRIPLE POINT

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ABSTRACT. We show that if a non-orientable surface embedded in 4-space has a projection into 3-space with at most one triple point, then it is ambient isotopic to a connected sum of some unknotted projective planes and an embedded surface in 4-space with vanishing normal Euler number.

1. INTRODUCTION

In [4] Kinoshita constructed infinitely many projective planes embedded in $\mathbf{R}^4$ which are not ambient isotopic to each other. Each of his projective planes is a connected sum of an unknotted projective plane and a 2-knot. Price and Roseman gave a method to construct a projective plane in $\mathbf{R}^4$ from an invertible 1-knot and proved that every spun projective plane is ambient isotopic to a connected sum of an unknotted projective plane and a 2-twist spun 2-knot [6]. It is unknown whether there is an example of a projective plane in $\mathbf{R}^4$ which is not a connected sum of an unknotted projective plane and a 2-knot.

On the other hand, Yoshikawa gave infinitely many Klein bottles in $\mathbf{R}^4$ each of which is not ambient isotopic to a connected sum of an unknotted projective plane and a (possibly knotted) projective plane [7].

In this paper we discuss the problem of when a non-orientable surface in $\mathbf{R}^4$ is ambient isotopic to a connected sum of unknotted projective planes and a surface in $\mathbf{R}^4$ from the viewpoint of *broken surface diagrams* due to Carter and Saito [2].

A *surface in* $\mathbf{R}^4$ means a closed and connected PL 2-manifold without boundary embedded in $\mathbf{R}^4$ locally flatly. We call a surface in $\mathbf{R}^4$ a *2-knot* if the embedded surface is homeomorphic to a 2-sphere. Moreover we can assume that the image $p(F)$ under the projection $p : \mathbf{R}^4 \longrightarrow \mathbf{R}^3$ is a *generic* surface; that is, the singular set $\Gamma(p(F))$ on $p(F)$ consists of (possibly empty) isolated branch points, double point curves, and isolated triple points. Then we prove the following theorem.

Theorem 1.1. *Let F be a non-orientable surface in $\mathbf{R}^4$ and $e(F)$ its normal Euler number. If $\Gamma(p(F))$ contains at most one triple point, then F is ambient isotopic to a connected sum of $|e(F)|/2$ unknotted projective planes and a surface F' in $\mathbf{R}^4$ with $e(F') = 0$.*

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We notice that for Yoshikawa's Klein bottle Kb, $\Gamma(p(Kb))$ contains no triple point and $e(Kb)$ is equal to zero.

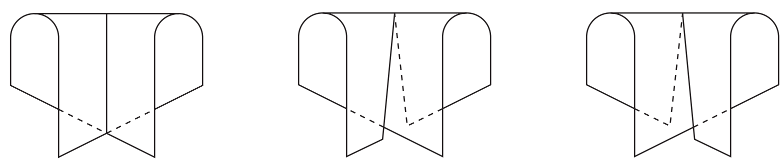
Let χ be the Euler characteristic of a non-orientable surface F in $\mathbf{R}^4$. Then the normal Euler number $e(F)$ is equal to $2\chi-4, 2\chi, \cdots, -2\chi$ or $4-2\chi$ (cf. [5]). Since the normal Euler number of every projective plane is ± 2, we have the following.

Corollary 1.2. *Let P be a projective plane in $\mathbf{R}^4$. If $\Gamma(p(P))$ contains at most one triple point, then P is ambient isotopic to a connected sum of an unknotted projective plane and a 2-knot.*

2. Projection with no triple point

Let F be a surface in $\mathbf{R}^4$ and $p(F)$ its generic projection, where $p : \mathbf{R}^4 \longrightarrow \mathbf{R}^3$ is the projection map.

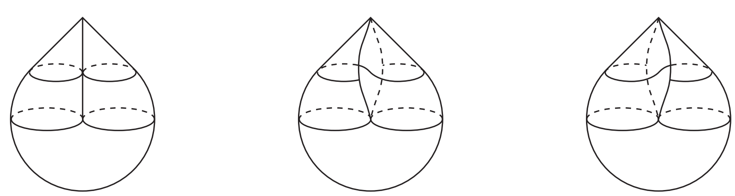
The regular neighborhood in $\mathbf{R}^3$ of a branch point on $p(F)$ is as depicted in Figure 1(A). We define signs of branch points in such a way that a branch point with the crossing information depicted in Figure 1(B) receives the sign $+1$, and one with the crossing information in Figure 1(C) receives the sign -1. Then the sum of signs taken over all the branch points in $\Gamma(p(F))$ is equal to the normal Euler number $e(F)$ of F (cf. [1]).



(A) (B) (C)

Figure 1.

A projective plane P in $\mathbf{R}^4$ is *unknotted* if it is ambient isotopic to the surface whose projection into $\mathbf{R}^3$ is as depicted in Figure 2(A). There are two distinct unknotted projective planes, one of which has the normal Euler number $e(P) = +2$ and the other has $e(P) = -2$; see Figure 2(B) and (C). We denote these P by P_+ (positive type) and P_- (negative type) respectively.



(A) (B) (C)

Figure 2.

Let $C_1, \cdots, C_n$ be the connected components of the singular set $\Gamma(p(F))$. Each C_i is regarded as a union of immersed loops and immersed arcs in $\mathbf{R}^3$ such that the endpoints of the immersed arcs are branch points. Suppose that C_i is one simple arc. Such a simple arc is called an *m-arc* (resp. *a-arc*) if the two branch points of its ends have the same sign (resp. opposite signs). Notice that the neighborhood of an m-arc (resp. a-arc) is homeomorphic to a Möbius band (resp. an annulus). Call an m-arc *positive* if it terminates with both positive type, or *negative* if it terminates with negative type. These terms are used in [3].

Lemma 2.1. *Let F be a non-orientable surface in $\mathbf{R}^4$. If $\Gamma(p(F))$ contains no triple point, then F is ambient isotopic to a connected sum of $|e(F)|/2$ unknotted projective planes and a surface F' in $\mathbf{R}^4$ with $e(F') = 0$.*

Proof. It is sufficient to prove the lemma in the case that $e(F) > 0$. Since $\Gamma(P(F))$ contains no triple point, $\Gamma(P(F))$ is a disjoint union of simple loops, a-arcs and m-arcs. Notice that $e(F) = 2(p - n)$, where p is the number of positive m-arcs and n is the number of negative m-arcs in $\Gamma(p(F))$. Then $\Gamma(p(F))$ contains at least $e(F)/2$ positive m-arcs, and so F is ambient isotopic to a connected sum of $e(F)/2$ P_+ and a surface F' in $\mathbf{R}^4$. Since $\Gamma(p(F'))$ is obtained from $\Gamma(p(F))$ by deleting the $e(F)/2$ positive m-arcs, we have $e(F') = 0$. □

3. Projection with one triple point

Let F be a surface in $\mathbf{R}^4$ and $p(F)$ its generic projection where $p : \mathbf{R}^4 \longrightarrow \mathbf{R}^3$ is a projection map. The neighborhood of a triple point on $p(F)$ consists of three sheets. These sheets can be labeled *top, middle* and *bottom*, and these indicate the relative position of the sheets with respect to the projection direction.

The following is the key lemma in this paper.

Lemma 3.1. (1) *Let S be a generic closed surface in $\mathbf{R}^3$ and c a simple closed curve in $\mathbf{R}^3$ such that c is transverse to S with $c \cap \Gamma(S) = \varnothing$. Then the number of $c \cap S$ is even.*

(2) *Let t be a triple point in $\Gamma(p(F))$ and H one of the three sheets around t. Suppose that there exists a subarc $\alpha : [0,1] \longrightarrow \Gamma(p(F))$ such that $\alpha(0) = \alpha(1) = t$, such that $\alpha([0,\varepsilon]) \subset H$, $\alpha([1-\varepsilon, 1]) \subset H$ for a small $\varepsilon > 0$, and such that $\alpha(int[0,1])$ contains no triple point. Then H is a top sheet or a bottom sheet.*

Proof. (1) Since the $\mathbf{Z}_2$-intersection number of c and S in $\mathbf{R}^3$ is zero, the number of $c \cap S$ is even.

(2) There are two cases as depicted in Figure 3, where the shaded sheet represents the sheet H. Suppose that H is a middle sheet in each case. Then its broken surface diagram is as depicted in Figure 4. Therefore the pasting map in the figure must be a rotation by 0° or 180° in Figure 4(A) and ±90° in Figure 4(B). This contradicts (1). □

(A) (B)

Figure 3.

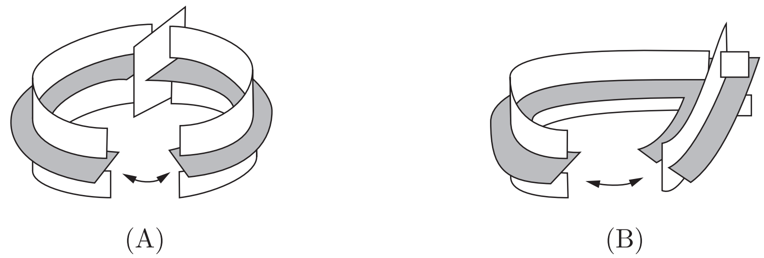


FIGURE 4.

We notice that in Lemma 3.1(2), even if H is a top sheet or a bottom sheet, the case in Figure 4(A) does not occur.

The following lemma is proved by performing a sequence of two Roseman moves — "a sheet moving through a saddle point" and "passing a branch point through a third sheet", where the *Roseman moves* are generalizations of the Reidemeister moves and are sufficient to move embedded surfaces in $\mathbf{R}^4$. The list of the Roseman moves appears in [2].

Lemma 3.2. *Let t be a triple point in $\Gamma(p(F))$ and H_i $(i = 1, 2, 3)$ three sheets around t. Suppose that the small arc $H_2 \cap H_3$ is extended to an immersed arc γ in $\Gamma(p(F))$ which ends in two branch points b and b' and that the subarc γ' of γ connecting between t and b contains no triple point inside. If H_1 is a top sheet or a bottom sheet, then the triple point t can be removed from $p(F)$ by deforming F by an ambient isotopy of $\mathbf{R}^4$ without introducing new triple points.*

Proof. Let D be a small neighborhood of t in H_1. Push D along γ' towards b. Since H_1 is a top sheet or a bottom sheet, we can push out D over b. This deformation can be realized by an ambient isotopy of $\mathbf{R}^4$ and does not introduce new triple points in the projection. See Figure 5 (the figure is in the case that H_1 is a top sheet). □

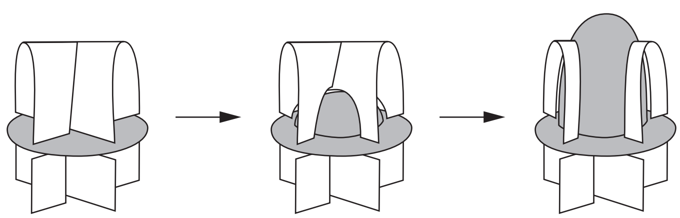
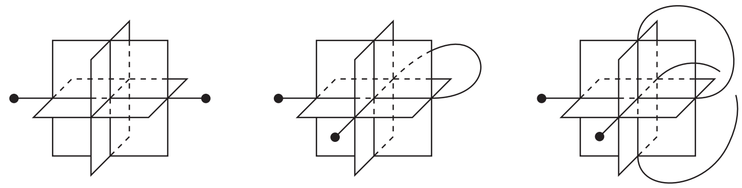


FIGURE 5.

Proof of Theorem 1.1. By Lemma 2.1, it is sufficient to consider the case that $\Gamma(p(F))$ contains just one triple point t. Let $C_1, \cdots, C_n$ be the connected components of $\Gamma(p(F))$ with $t \in C_1$.

If C_1 is a union of immersed loops, then all the branch points in $\Gamma(p(F))$ belong to $C_2 \cup \cdots \cup C_n$, where each of $C_2, \cdots, C_n$ is a simple loop or a simple arc. Then we can apply an argument similar to the argument in the proof of Lemma 2.1.

Suppose that C_1 contains at least one immersed arc, say γ. We will prove that t can be removed from C_1 without introducing new triple points. Let $c : [0, 1] \longrightarrow \mathbf{R}^3$



(A) (B) (C)

FIGURE 6.

be an immersion representing γ. First consider the case that $c^{-1}(t)$ consists of two or three points; see Figure 6(B) and (C). Then we can perform the deformation in Lemma 3.2 so that t can be removed from C_1. Consider the case that $c^{-1}(t)$ is one; see Figure 6(A). If the perpendicular sheet to γ is top or bottom, then we can also perform the deformation in Lemma 3.2. If the perpendicular sheet is middle, then C_1 has the only form which consists of three arcs intersecting at t by Lemma 3.1. Therefore we can perform the deformation in Lemma 3.2. $\square$

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