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SECTIONAL BODIES ASSOCIATED WITH A CONVEX BODY

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ABSTRACT. We define the sectional bodies associated to a convex body in $\mathbb{R}^n$ and two related measures of symmetry. These definitions extend those of Grünbaum (1963). As Grünbaum conjectured, we prove that the simplices are the most dissymmetrical convex bodies with respect to these measures. In the case when the convex body has a sufficiently smooth boundary, we investigate some limit behaviours of the volume of the sectional bodies.

INTRODUCTION

Let $\mathcal{K}^n$ be the set of convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff distance. For $K \in \mathcal{K}^n$, we denote by $|K|$ its volume relative to its affine hull and by g_K its centroid. Let S^{n-1} be the Euclidean sphere. For $1 \leq k \leq n-1$, let $\mathcal{G}_{n,k}$ be the Grassmann manifold of all k-dimensional vector subspaces of $\mathbb{R}^n$.

Recently, some authors described the limit behaviour of the volume of special bodies, or family of bodies, associated to a convex body K in $\mathbb{R}^n$, like the convex floating body $K_\delta := \{x \in K\,;\ \forall u \in S^{n-1}\ |\{y \in K\,; \langle y-x, u\rangle \geq 0\}| \geq \delta\}$ in [SW], the illumination body in [W], the Santaló regions in [MW], and the convolution body in [Sch]. In each case, they recovered the affine surface area $\Omega(K) := \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu(x)$, where for x in ∂K, the boundary of K, $\kappa(x)$ is the Gaussian curvature of K at x and μ denotes the Hausdorff measure. For a survey on $\Omega(K)$, we refer to Lutwak ([L]).

In this paper, in connection with K_δ, we introduce a new family of convex bodies, the *sectional bodies* $K(t) = \{x \in K\,;\ \forall u \in S^{n-1}\ |\{y \in K\,;\ \langle y-x, u\rangle = 0\}| \geq t\}$, for $t \geq 0$ and we study the limit behaviour of their volume. We prove that if K has positive curvature and C^2 boundary, then

$$\lim_{t\to 0} \frac{|K| - |K(t)|}{t^{\frac{2}{n-1}}} = c_n \int_{\partial K} \kappa(x)^{\frac{1}{n-1}} d\mu(x).$$

More generally, for $\phi : \mathcal{K}^n \times \mathcal{G}_{n,k} \to \mathbb{R}$ and $t \geq 0$, the $(\phi,\, k)$-*sectional bodies* of K are

$$K_{\phi,k}(t) := \{x \in K\,;\ \forall E \in \mathcal{G}_{n,k}\ |K \cap (x+E)| \geq t\phi(K,E)\}\,.$$

For the functions $\phi(K,E) = 1$, $\phi(K,E) = g(K,E) := |K \cap (g_K + E)|$ and $\phi(K,E) = m(K,E) := \max_{y\in K} |K \cap (y+E)|$, we respectively define $K_k(t)$, $K_{g,k}(t)$ and $K_{m,k}(t)$. For $k = n-1$, we reduce notation to $K_\phi(t)$, $K(t)$, $K_g(t)$ and $K_m(t)$. Let

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$f_{m,k}(K) = \max\{t > 0\,;\ K_{m,k}(t) \neq \emptyset\}$ and $g_{m,k}(K) = \max\{t > 0\,;\ g_K \in K_{m,k}(t)\}$. The family of bodies $K_{m,1}(t)$ and the derived measures $f_{m,1}$ and $g_{m,1}$ were introduced by Grünbaum in [G].

In the first part of this paper, we study the convexity and affine invariance properties of the sectional bodies and we relate them to the intersection and cross-section bodies. Then we prove that $f_{m,k}$ and $g_{m,k}$ are measures of symmetry for all $1 \leq k \leq n-1$ and, confirming a conjecture stated by Grünbaum in the case $k = 1$ (in [G], p. 254), we show that the simplices are among the most dissymmetrical convex bodies with respect to these measures. In the second part, we study the behaviour of the volume of $K_{\phi,k}(t)$ when t tends to 0. With some regularity assumptions on the convex K and the function ϕ, we prove that

$$\lim_{t\to 0} \frac{|K| - |K_\phi(t)|}{t^{\frac{2}{n-1}}} = \frac{v_{n-1}^{-\frac{2}{n-1}}}{2} \int_{\partial K} \phi(K, N(x))^{\frac{2}{n-1}} \kappa(x)^{\frac{1}{n-1}}\, d\mu(x),$$

where $N(x)$ is the unit normal vector to ∂K at x and v_{n-1} is the volume of the Euclidean ball in $\mathbb{R}^{n-1}$.

1. General properties

Following the notation of Grünbaum ([G]), we recall that a continuous function $f : \mathcal{K}^n \to [0,1]$ is an *affine invariant measure of symmetry* if it satisfies $f(AK) = f(K)$ for every $K \in \mathcal{K}_n$ and every nonsingular affine transformation A and $f(K) = 1$ if and only if K is symmetric. An application $F : \mathcal{K}^n \to \mathcal{K}^n$ is *affine invariant* if it is lower semi-continuous and satisfies $F(AK) = AF(K)$ for every $K \in \mathcal{K}^n$ and every non-singular affine transform A. For $x \in K$ and $1 \leq k \leq n-1$, let

$$f_{\phi,k}(x, K) = \min_{E\in\mathcal{G}_{n,k}} \frac{|K \cap (x + E)|}{\phi(K, E)}, \quad f_{\phi,k}(K) = \max_{x\in K} f_{\phi,k}(x, K),$$

and $g_{\phi,k}(K) = f_{\phi,k}(g_K, K)$. For $k = 1$ and $\phi = m$, these definitions were introduced by Grünbaum. The boundary of the sectional body $K_{\phi,k}(t)$ of K is a level set of $f_{\phi,k}(x, K)$, i.e. $K_{\phi,k}(t) = \{x \in K\,;\ f_{\phi,k}(x, K) \geq t\}$. Hence as noticed in the introduction, one has $f_{\phi,k}(K) = \max\{t > 0\,;\ K_{\phi,k}(t) \neq \emptyset\}$ and $g_{\phi,k}(K) = \max\{t > 0\,;\ g_K \in K_{\phi,k}(t)\}$. The set $C_{\phi,k}(K) := K_{\phi,k}(f_{\phi,k}(K))$ is called the critical set of K.

1.1. **Convexity.** It follows from the theorem of Brunn-Minkowski that $x \mapsto f_{\phi,k}(x, K)^{\frac{1}{k}}$ is concave on K. Hence for $0 \leq \lambda \leq 1$ and $0 \leq t_1, t_2 < f_{\phi,k}(K)$,

$$\lambda K_{\phi,k}(t_1^k) + (1-\lambda)K_{\phi,k}(t_2^k) \subset K_{\phi,k}\Big(\big(\lambda t_1 + (1-\lambda)t_2\big)^k\Big).$$

In particular, for all $0 \leq t < f_{\phi,k}(K)$, $K_{\phi,k}(t)$ is a convex body. It also follows that $t \mapsto |K_{\phi,k}(t^k)|^{\frac{1}{n}}$ is concave on $[0, f_{\phi,k}(K)]$.

Let us prove that $t \mapsto K_{\phi,k}(t)$ and $t \mapsto |K_{\phi,k}(t)|$ are decreasing on $[0, f_{\phi,k}(K)]$: The concavity of the function $x \mapsto f_{\phi,k}(x, K)^{\frac{1}{k}}$ implies its continuity and we get

$$\begin{aligned} |K_{\phi,k}(t_1)| - |K_{\phi,k}(t_2)| &= |\{x \in K\,;\ t_1 \leq f_{\phi,k}(x, K) < t_2\}| \\ &\neq 0, \qquad 0 < t_1 < t_2 \leq f_{\phi,k}(K). \end{aligned}$$

Moreover, if x is an exposed point of ∂K, then $f_{\phi,k}(x, K) = 0$. Hence for all $t > 0$, $K(t) \neq K(0) = K$. From the continuity of $x \mapsto f_{\phi,k}(x, K)$, we get $|K(t)| \neq |K|$.

1.2. **Affine invariance.** Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a nonsingular linear transformation, let $1 \le k \le n-1$ and $E \in \mathcal{G}_{n,k}$ endowed with the Euclidean structure induced by the one of $\mathbb{R}^n$. Let $T_E : E \to \mathbb{R}^n$, satisfying $T_E x = Tx$ for every $x \in E$, and let $D_E(T) := (\det(T_E^* T_E))^{\frac{1}{2}}$; then it is well known that for all convex body $C \subset E$, we have $|T(C)| = D_E(T)|C|$.

Let $\phi : \mathcal{K}^n \times \mathcal{G}_{n,k} \to \mathbb{R}$, continuous, such that for all $(K,E) \in \mathcal{K}^n \times \mathcal{G}_{n,k}$ and for all nonsingular affine transformation A on $\mathbb{R}^n$, with $A(x) = T(x)+z$, where $z \in \mathbb{R}^n$, we have $\phi(AK, AE) = D_E(T)\phi(K,E)$. Then the function $(x,K) \mapsto f_{\phi,k}(x,K)$ is continuous on $\{(x,K) \in \mathbb{R}^n \times \mathcal{K}^n\,;\ x \in \mathrm{int}(K)\}$; hence for all $t > 0$ it is easy to see that $K \mapsto K_{\phi,k}(t)$ is continuous. For all $x \in AK$, $f_{\phi,k}(x, AK) = f_{\phi,k}(A^{-1}x, K)$. Hence for $t \ge 0$, $A(K_{\phi,k}(t)) = (AK)_{\phi,k}(t)$; thus $K \mapsto K_{\phi,k}(t)$ is affine invariant.

We deduce that for all $1 \le k \le n-1$ and $t \ge 0$, $K \mapsto K_{g,k}(t)$ and $K \mapsto K_{m,k}(t)$ are affine invariant. Notice that $K \mapsto K_k(t)$ is generally not invariant, but is continuous.

1.3. **Relationship with the intersection and cross-section bodies.** For $x \in K \in \mathcal{K}^n$, the *$x$-intersection body* of K, I_xK and the *cross-section body*, CK are defined by their radial functions

$$\rho_{I_xK}(u) = |K \cap (x + u^{\perp})| \quad \text{and } \rho_{CK}(u) = \max_{y\in K} |K \cap (y + u^{\perp})|,$$

for all $u \in S^{n-1}$. For all $x \in K$, we have $I_xK \subset CK$ and $\bigcup_{x\in K} I_xK = CK$. Moreover, these bodies are related to the sectional bodies: one has $K(t) = \{x \in K\,;\ I_xK \supset tB\}$, $K_g(t) = \{x \in K\,;\ I_xK \supset t\, I_{g_K}K\}$ and $K_m(t) = \{x \in K\,;\ I_xK \supset t\, CK\}$. It follows from [MM] that $\partial I_xK \cap \partial CK \neq \emptyset$ for all $x \in K$. Since I_xK and CK are symmetric, we get $\inf\{b\,;\ I_xK \subset b\, CK\} = 1$.

With the following distance on the set of centrally symmetric convex bodies, $\mathcal{K}_n^0$, $d(K,L) = \inf\{b/a\,;\ aK \subset L \subset bK\}$, for K and L in $\mathcal{K}_n^0$, one has, for all $x \in K$, $f_m(x,K) = d(I_xK, CK)^{-1}$,

$$g_m(K) = d(I_{g_K}K, CK)^{-1} \quad \text{and} \quad f_m(K) = \Big(\inf_{x\in K} d(I_xK, CK)\Big)^{-1}.$$

For $k = 1$ instead of $n-1$, the same relationship can be obtained for the *x-chordal symmetral* of K, $\tilde{\Delta}_xK$ and the *difference body* of K, DK defined by their radial function: $\rho_{\tilde{\Delta}_xK}(u) = |K \cap (x + \mathbb{R}u)|$ and $\rho_{DK}(u) = \max_{y\in K} |K \cap (y + \mathbb{R}u)| = \rho_{K-K}(u)$, for all $u \in S^{n-1}$. Indeed, one has $f_{m,1}(x,K) = d(\tilde{\Delta}_xK, DK)^{-1}$,

$$g_{m,1}(K) = d(\tilde{\Delta}_{g_K}K, DK)^{-1} \quad \text{and} \quad f_{m,1}(K) = \Big(\inf_{x\in K} d(\tilde{\Delta}_xK, DK)\Big)^{-1},$$

for all $x \in K$. See [Ga] for more results on these bodies.

1.4. **The maximal sectional measures of symmetry.** The following result was proved by Kovetz ([K]) in the case $k = 1$.

Theorem 1. *For all* $1 \le k \le n-1$, $f_{m,k}(K) = \max_{x\in K} \min_{E\in\mathcal{G}_{n,k}} \dfrac{|K \cap (x+E)|}{\max_{y\in K} |K \cap (y+E)|}$ *and* $g_{m,k}(K) = \min_{E\in\mathcal{G}_{n,k}} \dfrac{|K \cap (g_K+E)|}{\max_{y\in K} |K \cap (y+E)|}$ *are affine invariant measures of symmetry.*

Proof. The only point which needs to be checked precisely is the following. Let $K \in \mathcal{K}^n$ be such that $f_{m,k}(K) = 1$. Then there exists $x \in K$, such that for all $E \in \mathcal{G}_{n,k}$, $|K \cap (x+E)| = \max_{y \in K} |K \cap (y+E)|$. By affine invariance, we may assume that $x = 0$. Let $F \in \mathcal{G}_{n,k+1}$ be fixed. Considering only the k-dimensional subspaces $E \subset F$, we obtain that all the hyperplane sections of $K \cap F$ through the origin are the sections of maximal volume among the sections by parallel hyperplanes. From [MMO], this implies that $K \cap F$ is centrally symmetric. Since this is true for all $F \in \mathcal{G}_{n,k+1}$, K is centrally symmetric. □

Now we are interested in a lower bound for this measure of symmetry.

Theorem 2. *For all* $1 \le k \le n-1$ *and for all convex body* $K \subset \mathbb{R}^n$*, one has*

$$f_{m,k}(K) \ge g_{m,k}(K) \ge g_{m,k}(\Delta) = f_{m,k}(\Delta) = \left(\frac{k+1}{n+1}\right)^k,$$

where Δ *is any simplex in* $\mathbb{R}^n$.

Proof. From [F], for any $K \in \mathcal{K}^n$, one has $g_{m,k}(K) \ge (\frac{k+1}{n+1})^k = g_{m,k}(\Delta)$. Hence, it is enough to prove that $f_{m,k}(\Delta) = g_{m,k}(\Delta)$. This is the same as $g_\Delta \in C_{m,k}(\Delta)$. First notice that the critical set of any convex body is convex, affine invariant and has empty interior. Suppose that there is $x \in C_{m,k}(\Delta)$, $x \ne g_\Delta$. Then the convex hull of the set of images of x, under the group of affine maps of Δ onto itself, which leave only g_Δ fixed, has non-empty interior, which is absurd. Since $C_{m,k}(\Delta) \ne \emptyset$, it follows that $C_{m,k}(\Delta) = \{g_\Delta\}$. □

Remark 1. We conjecture that for any $1 \le k \le n-1$, the simplices are the only convex bodies satisfying $f_{m,k}(K) = (\frac{k+1}{n+1})^k$.

2. Sectional bodies and Gaussian curvature

2.1. **Results.** The Euclidean ball of center x and radius r in $\mathbb{R}^n$ is denoted by $B(x,r)$, and the Euclidean norm by $|\cdot|$. Let K be a convex body in $\mathbb{R}^n$ with C^2 boundary and positive curvature. For all $x \in \partial K$, denote by $N(x)$ the unit normal vector to ∂K at x; let T_x be the tangent hyperplane of K at x and $S_x : \mathbb{R}^n \to \mathbb{R}^n$ the composition of the rotation U_x and the translation of vector x such that $U_x(0,...,0,1) = -N(x)$ and which maps the $n-1$ first coordinates of $\mathbb{R}^n$ onto T_x. We denote by $\varphi_x : \mathbb{R}^{n-1} \to \mathbb{R}$ the strictly convex mapping which satisfies that, for some $s_0 > 0$,

$$K \cap \{z \in \mathbb{R}^n \,;\ \langle z - x, -N(x)\rangle \le s_0\} = S_x(\{(z,s) \in \mathbb{R}^{n-1} \times \mathbb{R} \,;\ \varphi_x(z) \le s \le s_0\}).$$

The quadratic form $d^2\varphi_x(0)$ is positive, its eigenvalues $(k_i(x))_{1 \le i \le n-1}$ are *the principal curvatures* of K at x and the *Gaussian curvature* of K at x is $\kappa(x) = \prod_{i=1}^{n-1} k_i(x)$. We refer to [S] for more intrinsic definitions and results on the curvature. In the following, we denote $c_n := \frac{1}{2} v_{n-1}^{-2/(n-1)}$.

Theorem 3. *Let* $\phi : \mathcal{K}^n \times S^{n-1} \to \mathbb{R}$ *and let* $K \in \mathcal{K}^n$ *with* C^2 *boundary and positive curvature. If* $u \mapsto \phi(K,u)$ *is even, continuous, positive and bounded on* S^{n-1}*, then*

$$\lim_{t \to 0} \frac{|K| - |K_\phi(t)|}{t^{\frac{2}{n-1}}} = c_n \int_{\partial K} \phi(K, N(x))^{\frac{2}{n-1}} \kappa(x)^{\frac{1}{n-1}} \, d\mu(x) \,.$$

As an immediate corollary, we get

Corollary 1. *Let $K \in \mathcal{K}^n$ with C^2 boundary and positive curvature. Then*

$$\begin{aligned}
\lim_{t\to 0} \frac{|K|-|K(t)|}{t^{\frac{2}{n-1}}} &= c_n \int_{\partial K} \kappa(x)^{\frac{1}{n-1}}\, d\mu(x)\,, \\
\lim_{t\to 0} \frac{|K|-|K_g(t)|}{t^{\frac{2}{n-1}}} &= c_n \int_{\partial K} |K\cap (g_K + N(x)^{\perp})|^{\frac{2}{n-1}} \kappa(x)^{\frac{1}{n-1}}\, d\mu(x)\,, \\
\lim_{t\to 0} \frac{|K|-|K_m(t)|}{t^{\frac{2}{n-1}}} &= c_n \int_{\partial K} \max_{y\in K} |K\cap (y + N(x)^{\perp})|^{\frac{2}{n-1}} \kappa(x)^{\frac{1}{n-1}}\, d\mu(x)\,.
\end{aligned}$$

We also find the equivalent of $|K_1(t)|$:

Theorem 4. *Let K be a convex body in $\mathbb{R}^n$ with C^2 boundary and positive curvature. Let $k_1(x)$ be the maximum of the principal curvatures of K at $x \in \partial K$. Then*

$$\lim_{t\to 0} \frac{|K|-|K_1(t)|}{t^2} = \frac{1}{8}\int_{\partial K} k_1(x) d\mu(x)\,.$$

Remarks. 1) Using the change of variable $N : \partial K \to S^{n-1}$, the quantities appearing as limits in Theorem 3 and Corollary 1 can be expressed as integrals over S^{n-1}. If we denote by $\pi(u)$ the product of the $n-1$ principal radii of curvature of K in the direction u, i.e. $\pi(N(x)) = \kappa(x)^{-1}$ for all $x \in \partial K$, we have

$$S_\phi(K) := \int_{\partial K} \phi(K, N(x))^{\frac{2}{n-1}} \kappa(x)^{\frac{1}{n-1}}\, d\mu(x) = \int_{S^{n-1}} \phi(K,u)^{\frac{2}{n-1}} \pi(u)^{\frac{n-2}{n-1}}\, d\mu(u)\,.$$

2) Since $K \mapsto K_g(t)$ and $K \mapsto K_m(t)$ are affinely invariant, $S_g(K)$ and $S_m(K)$ are invariant under special affine transformation, like the affine surface area defined by

$$\Omega(K) := \int_{\partial K} \kappa(x)^{\frac{1}{n+1}}\, d\mu(x) = \int_{S^{n-1}} \pi(u)^{\frac{n}{n+1}}\, d\mu(u)\,.$$

Using Hölder's inequality, we see that $S_g(K)$ (respectively $S_m(K)$) is related to $\Omega(K)$ and the volume of the intersection body $I_{g_K}K$ (resp. of the cross-section body CK):

$$S_g(K)^{n(n-1)} \le (n|I_{g_K}K|)^2 \Omega(K)^{(n+1)(n-2)}$$

and

$$S_m(K)^{n(n-1)} \le (n|CK|)^2 \Omega(K)^{(n+1)(n-2)}.$$

3) For a non-constant function ϕ, it is easy to see that one cannot generalize Theorem 3 to lower dimensional sections. But for $\phi = 1$, we conjecture that the following, proved for $j = 1$ and $n-1$, still holds for all $1 \le j \le n-1$: if $k_1(x) \ge \dots \ge k_{n-1}(x)$ are the principal curvature at $X \in \partial K$, one has

$$\lim_{t\to 0} \frac{|K|-|K_j(t)|}{t^{2/j}} = c_{j+1} \int_{\partial K} \prod_{i=1}^{j} k_i(x)^{1/j}\, d\mu(x)\,.$$

2.2. **Proofs.** We start with some considerations which will be used in the proofs of both theorems. With the preceding notations, for $z = (z_1, ..., z_{n-1})$ in the basis of eigenvectors of $d^2\varphi_x(0)$ in $\mathbb{R}^{n-1}$, one has $\varphi_x(z) = \sum_{i=1}^{n-1} \frac{k_i(x)}{2} z_i^2 + |z|^2\eta_x(|z|)$, where $\lim_{s\to 0} \eta_x(s) = 0$. For all $x \in \partial K$, ϕ_x is C^2. Hence, by compactness of ∂K, the function $\eta = \sup_{x\in K} |\eta_x|$ still satisfies $\lim_{s\to 0} \eta(s) = 0$. For fixed $\varepsilon > 0$, there exists $\alpha > 0$, such that

$$(1-\varepsilon)\sum_{i=1}^{n-1} \frac{k_i(x)}{2} z_i^2 \le \varphi_x(z) \le (1+\varepsilon)\sum_{i=1}^{n-1} \frac{k_i(x)}{2} z_i^2 \quad \text{for all } \mid z \mid < \alpha \text{ and } x \in \partial K.$$

For a fixed $x \in \partial K$, let $P_\varepsilon = \{(z, s) \in \mathbb{R}^{n-1} \times \mathbb{R};\ s \ge (1+\varepsilon)\sum_{i=1}^{n-1} k_i(x) z_i^2/2\}$. There exists s_1 such that, if $H_s = \{(z, s);\ z \in \mathbb{R}^{n-1}\}$, we get the following inclusion, known as the Dupin's lemma:

$$(1) \qquad P_\varepsilon \cap H_s \subset S_x^{-1}(K) \cap H_s \subset P_{-\varepsilon} \cap H_s \quad \text{for all} \quad s \le s_1\,.$$

Let $\phi : \mathcal{K}^n \times \mathcal{G}_{n,k} \to \mathbb{R}$ satisfy that $E \mapsto \phi(K, E)$ is continuous, positive and bounded on $\mathcal{G}_{n,k}$. Since $K_{\phi,k}(t)$ is invariant by translation of K, we may assume that $0 \in C_{\phi,k}(K) \subset K_{\phi,k}(t)$, for all $0 \le t \le f_{\phi,k}(K)$. It is well known that

$$(2) \qquad |K| - |K_{\phi,k}(t)| = \frac{1}{n}\int_{\partial K} \langle x, N(x)\rangle \left(1 - \rho_{K_{\phi,k}(t)}(x)^n\right) d\mu(x)\,.$$

In the following, we will denote $t_0 := f_{\phi,k}(K)$ and consider $t \in [0, t_0[$, hence $K_{\phi,k}(t)$ will have non-empty interior. For all $x \in \partial K$, we define $\lambda_t(x) := \rho_{K_{\phi,k}(t)}(x)$. It satisfies $\min_{E\in\mathcal{G}_{n,k}} \dfrac{|K \cap (\lambda_t x + E)|}{\phi(K, E)} = t$ and $\partial K_{\phi,k}(t) = \{\lambda_t(x)x;\ x \in \partial K\}$. To prove Theorems 3 and 4, we first need some lemmas.

Lemma 1. *Let $K \in \mathcal{K}^n$ with C^2 boundary and positive curvature. Let $1 \le k \le n-1$ and $\phi : \mathcal{K}^n \times \mathcal{G}_{n,k} \to \mathbb{R}$, such that $E \mapsto \phi(K, E)$ is continuous, positive and bounded on $\mathcal{G}_{n,k}$. Then there exist $r > 0$ and $\alpha > 0$ such that for all $t < \alpha$ and $x \in \partial K$,*

$$\frac{1}{n}\langle x, N(x)\rangle \frac{1}{t^{2/k}}\left(1 - \rho_{K_{\phi,k}(t)}(x)^n\right) \le \frac{v_k^{-2/k}}{r}\,.$$

Proof. For $x \in \partial K$, $t \mapsto \lambda_t(x)$ is continuous and decreasing from $[0, t_0]$ onto $[\lambda_{t_0}(x), 1]$ and for $t \in [0, t_0[$, $x \mapsto \lambda_t(x)$ is continuous on ∂K. Hence for all sequence (t_n) decreasing to 0, the sequence $(\lambda_{t_n}(x))$ is increasing, continuous on the compact ∂K and converges pointwise to 1 when n grows to infinity. From Dini's theorem, we deduce that $(\lambda_{t_n}(x))$ converges uniformly to 1. Therefore for all $\varepsilon > 0$, there exists α such that, for all $t < \alpha$ and for all $x \in \partial K$, we have $0 \le 1 - \lambda_t(x) < \varepsilon$.

Since K has positive curvature, there exists $r > 0$ such that, for all $x \in \partial K$, $B_{x,r} := B(x - rN(x), r) \subset K$. Let $r_2 \ge r_1 > 0$ satisfying $r_1 B \subset K \subset r_2 B$. Then $K^* \subset B/r_1$ and $r_1 \le ||N(x)||_{K^*} = h_K(N(x)) = \langle x, N(x)\rangle \le r_2$, thus $\frac{r\langle x, N(x)\rangle}{|x|^2} \ge \frac{r r_1}{r_2^2}$. Hence there is $\alpha > 0$ such that, for all $t < \alpha$ and for all $x \in \partial K$, we have

$$(3) \qquad 0 \le 1 - \lambda_t(x) \le \frac{r\langle x, N(x)\rangle}{|x|^2}\,.$$

For $t < \alpha$ and fixed $x \in \partial K$, let $\lambda_t = \lambda_t(x)$. Then

$$\min_{E\in\mathcal{G}_{n,k}} \frac{|B_{x,r} \cap (\lambda_t x + E)|}{\phi(K, E)} \le \min_{E\in\mathcal{G}_{n,k}} \frac{|K \cap (\lambda_t x + E)|}{\phi(K, E)} = t\,.$$

Let $M = \max_{E\in\mathcal{G}_{n,k}} \phi(K,E)$. It is clear that for $y \in B := B(0,1)$, $\min_{E\in\mathcal{G}_{n,k}} |B\cap(y+E)|$ is achieved when $E\subset y^\perp$. Similarly we have

$$\begin{aligned}\min_{E\in\mathcal{G}_{n,k}} |B_{x,r}\cap(\lambda_t x+E)| &= v_k\Big|r^2 - |rN(x)-(1-\lambda_t)x|^2\Big|^{\frac{k}{2}}\\ &= v_k\Big|2(1-\lambda_t)r\langle x,N(x)\rangle - (1-\lambda_t)^2|x|^2\Big|^{\frac{k}{2}} \le Mt\,.\end{aligned}$$

Since $t<\alpha$, from (3) we get $2(1-\lambda_t)r\langle x,N(x)\rangle-(1-\lambda_t)^2|x|^2 \le (Mt/v_k)^{2/k}$. Hence, denoting $h=h_K(N(x))=\langle x,N(x)\rangle$,

$$1-\lambda_t \le \frac{rh}{|x|^2}\Big(1-\Big(1-\frac{|x|^2}{r^2h^2}\Big(\frac{Mt}{v_k}\Big)^{\frac{2}{k}}\Big)^{\frac12}\Big) \le \frac{1}{rh}\Big(\frac{Mt}{v_k}\Big)^{\frac{2}{k}}.$$

Finally for $t<\alpha$ and $x\in\partial K$

$$\frac1n\langle x,N(x)\rangle\Big(\frac{1-\lambda_t(x)^n}{t^{\frac2k}}\Big) \le \langle x,N(x)\rangle\Big(\frac{1-\lambda_t(x)}{t^{\frac2k}}\Big) \le \frac1r\Big(\frac{M}{v_k}\Big)^{\frac2k}. \qquad \square$$

Lemma 2. *Let $(e_1,\dots,e_n)$ be the canonical basis of $\mathbb{R}^n$, let Ψ be an even, continuous, positive function on S^{n-1} and let $P=\Big\{(x_1,\dots,x_n)\in\mathbb{R}^n\,;\ x_n\ge\sum_{i=1}^{n-1}\frac{k_i}{2}x_i^2\Big\}$. Then for all $y=(y_1,\dots,y_n)\in\mathbb{R}^n$ such that $y_n>0$, one has*

$$\min_{u\in S^{n-1}}\frac{|P\cap(\mu y+u^\perp)|}{\Psi(u)} \underset{\mu\to0}{\sim} \frac{|P\cap(\mu y+e_n^\perp)|}{\Psi(e_n)} = \frac{v_{n-1}}{\Psi(e_n)}\left(\frac{(2\mu y_n)^{n-1}}{\prod_{i=1}^{n-1}k_i}\right)^{\frac12}.$$

Proof. We may assume that $y\in P$, $\mu\le1$ and $u_n\ne0$. Hence, we replace $u\in S^{n-1}$ by $u+e_n$, with $u\in\mathbb{R}^{n-1}$, and we extend Ψ to $\mathbb{R}^n\backslash\{0\}$ by $\Psi(x)=\Psi(\frac{x}{|x|})$. We get

$$\min_{u\in S^{n-1}}\frac{|P\cap(\mu y+u^\perp)|}{\Psi(u)} = \min_{u\in\mathbb{R}^{n-1}}\frac{|P\cap(\mu y+(e_n+u)^\perp)|}{\Psi(u+e_n)}\,.$$

We have $P\cap(\mu y+(e_n+u)^\perp)=\Big\{x\in\mathbb{R}^n\,;\ \langle x-\mu y,e_n+u\rangle=0\ ,\ x_n\ge\sum_{i=1}^{n-1}\frac{k_i}{2}x_i^2\Big\}$. Thus $Q:=P\cap(\mu y+(e_n+u)^\perp)$ is an ellipsoid in the affine hyperplane of equation $\{x\in\mathbb{R}^n\,;\ x_n=\mu y_n-\sum_{i=1}^{n-1}(x_i-\mu y_i)u_i\}$. Hence, the projection of Q onto $e_n^\perp$ is

$$\Big\{(x_1,\dots,x_{n-1},0)\,;\ \sum_{i=1}^{n-1}k_i\Big(x_i+\frac{u_i}{k_i}\Big)^2\le 2\mu y_n+\sum_{i=1}^{n-1}\Big(\frac{u_i^2}{k_i}+2\mu y_iu_i\Big)\Big\}.$$

Therefore, if we define $f_\mu(u)=|Q|$, we get

$$f_\mu(u)=\frac{|e_n+u|}{\langle e_n,u+e_n\rangle}\times\frac{v_{n-1}}{(\prod_{i=1}^{n-1}k_i)^{\frac12}}\Big(2\mu y_n+\sum_{i=1}^{n-1}\Big(\frac{u_i^2}{k_i}+2\mu y_iu_i\Big)\Big)^{\frac{n-1}{2}}.$$

Let $g_\mu(u)=\dfrac{f_\mu(u)\Psi(e_n)}{f_\mu(0)\Psi(e_n+u)}$; then

$$g_\mu(u)=\frac{\Psi(e_n)}{\Psi(e_n+u)}\Big(1+\sum_{i=1}^{n-1}u_i^2\Big)^{\frac12}\Big(1+\frac{1}{2y_n}\sum_{i=1}^{n-1}\Big(\frac{1}{\mu k_i}(u_i+\mu k_iy_i)^2-\mu k_iy_i^2\Big)\Big)^{\frac{n-1}{2}}.$$

We want to prove that $\lim_{\mu\to 0}\min_{u\in\mathbb{R}^{n-1}} g_\mu(u) = 1$. It is clear that $\min_{u\in\mathbb{R}^{n-1}} g_\mu(u)\le 1$. On the other hand, for any fixed $\alpha>0$, one has $\lim_{\mu\to 0}\min_{|u|>\alpha} g_\mu(u) = +\infty$; hence

$$\lim_{\mu\to 0}\min_{u\in\mathbb{R}^{n-1}} g_\mu(u) = \lim_{\mu\to 0}\min_{|u|\le\alpha} g_\mu(u) \qquad \forall\alpha>0\,. \tag{4}$$

Moreover, for $|u|\le y_n/|y|$,

$$g_\mu(u)\ge \frac{\Psi(e_n)}{\Psi(e_n+u)}\Big(1+\frac{1}{y_n}\sum_{i=1}^{n-1} y_i u_i\Big)^{\frac{n-1}{2}} \ge \frac{\Psi(e_n)}{\Psi(e_n+u)}\Big(1-\frac{1}{y_n}|y||u|\Big)^{\frac{n-1}{2}}.$$

Hence by (4), we have $\lim_{\mu\to 0}\min_{u\in\mathbb{R}^{n-1}} g_\mu(u)\ge \Big(1-\dfrac{1}{y_n}|y|\alpha\Big)^{\frac{n-1}{2}}\min_{|u|\le\alpha}\dfrac{\Psi(e_n)}{\Psi(e_n+u)}$, for all $0<\alpha\le y_n/|y|$. When $\alpha\to 0$, by continuity of Ψ, we get $\lim_{\mu\to 0}\min_{u\in\mathbb{R}^{n-1}} g_\mu(u) = 1$. □

Lemma 3. *Let $K\in\mathcal{K}^n$ with C^2 boundary and positive curvature. Let ϕ such that $u\mapsto\phi(K,u)$ is even continuous, positive and bounded on S^{n-1}. Then for all $x\in\partial K$,*

$$\lim_{t\to 0}\frac{1}{n}\langle x,N(x)\rangle\frac{1}{t^{\frac{2}{n-1}}}\left(1-\rho_{K_\phi(t)}(x)^n\right) = c_n\phi(N(x))^{\frac{2}{n-1}}\kappa(x)^{\frac{1}{n-1}}.$$

Proof. We denote $\phi(u):=\phi(K,u)$.

1) We fix $x\in\partial K$; denote $\lambda_t=\lambda_t(x)$ and $s(t)=(1-\lambda_t)\langle x,N(x)\rangle$. Since $\lim_{t\to 0}\lambda_t = 1$, there exists t_1 such that $s(t)\le s_1$, for $t\le t_1$. Hence, from (1), one has

$$\begin{aligned} t=\min_{u\in S^{n-1}}\frac{|K\cap(\lambda_t x+u^\perp)|}{\phi(u)} &\le \frac{|K\cap(\lambda_t x+N(x)^\perp)|}{\phi(N(x))}\le\frac{|P_{-\varepsilon}\cap H_{s(t)}|}{\phi(N(x))}\\ &\le \frac{v_{n-1}}{\phi(N(x))}\Big(\frac{2s(t)}{1-\varepsilon}\Big)^{\frac{n-1}{2}}\prod_{i=1}^{n-1}k_i^{-\frac12}. \end{aligned}$$

Thus for $t\le t_1$, $\dfrac{s(t)}{t^{\frac{2}{n-1}}}\ge\dfrac{1-\varepsilon}{2}\Big(\dfrac{\phi(N(x))}{v_{n-1}}\Big)^{\frac{2}{n-1}}\kappa(x)^{\frac{1}{n-1}}$.

2) To prove the reverse inequality, it is more convenient to work with $S_x^{-1}(K)$ instead of K. Recall that $S_x = x+U_x$, where $U_x(e_n)=-N(x)$. Let $y=S_x^{-1}(0)$ and $\mu_t = 1-\lambda_t$. We define $H_s^+=\{(x,l)\in\mathbb{R}^{n-1}\times\mathbb{R}\,;\;0\le l\le s\}$ and $\Psi(u)=\phi(U_x(u))$. Using (1), we obtain for $t\le t_1$

$$\begin{aligned} t=\min_{u\in S^{n-1}}\frac{|K\cap(\lambda_t x+u^\perp)|}{\phi(u)} &= \min_{u\in S^{n-1}}\frac{|S_x^{-1}(K)\cap(\mu_t y+u^\perp)|}{\Psi(u)}\\ &\ge \min_{u\in S^{n-1}}\frac{|P_\varepsilon\cap H_{s_1}^+\cap(\mu_t y+u^\perp)|}{\Psi(u)}. \end{aligned}$$

Denote $C_t=\{u\in S^{n-1}\,;\;P_\varepsilon\cap H_{s_1}^-\cap(\mu_t y+u^\perp)\ne\emptyset\}$. It is clear that there exists $c>0$ such that for all $t\le t_1/2$, we have $\min_{u\in C_t}|P_\varepsilon\cap H_{s_1}^+\cap(\mu_t y+u^\perp)|\ge c$. Hence, using Lemma 2, one has

$$\begin{aligned} \min_{u\in S^{n-1}}\frac{|P_\varepsilon\cap H_{s_1}^+\cap(\mu_t y+u^\perp)|}{\Psi(u)} &\underset{t\to 0}{\sim}\min_{u\in S^{n-1}}\frac{|P_\varepsilon\cap(\mu_t y+u^\perp)|}{\Psi(u)}\\ &\underset{t\to 0}{\sim}\frac{v_{n-1}}{\Psi(e_n)}\Big(\frac{(2\mu_t y_n)^{n-1}}{\prod_{i=1}^{n-1}k_i}\Big)^{\frac12}. \end{aligned}$$

Thus for some $t_2 > 0$, one has $t \geq \dfrac{v_{n-1}}{(1+\varepsilon)\Psi(e_n)}\Big(\dfrac{2\mu_t y_n}{1+\varepsilon}\Big)^{\frac{n-1}{2}}\kappa(x)^{-\frac{1}{2}}$, for all $t \leq t_2$. This means that

$$\frac{(1-\lambda_t)\langle x, N(x)\rangle}{t^{\frac{2}{n-1}}} \leq \frac{(1+\varepsilon)^{\frac{n+1}{n-1}}}{2}\Big(\frac{\phi(N(x))}{v_{n-1}}\Big)^{\frac{2}{n-1}}\kappa(x)^{\frac{1}{n-1}}, \quad \text{for all } t \leq t_2.$$

Finally, we conclude that $\lim_{t\to 0} \dfrac{(1-\lambda_t)\langle x, N(x)\rangle}{t^{\frac{2}{n-1}}} = \dfrac{1}{2}\Big(\dfrac{\phi(N(x))}{v_{n-1}}\Big)^{\frac{2}{n-1}}\kappa(x)^{\frac{1}{n-1}}$. Since $\frac{1}{n}(1-\lambda_t^n) \underset{t\to 0}{\sim} 1-\lambda_t$, we obtain the result. □

Lemma 4. *Let $(e_i)_{1\leq i\leq n}$ be the canonical basis of $\mathbb{R}^n$ and $k_1 \geq ... \geq k_{n-1} > 0$. Let $P = \Big\{(x_1, ..., x_n) \in \mathbb{R}^n \ ; x_n \geq \sum_{i=1}^{n-1} \dfrac{k_i}{2} x_i^2\Big\}$. Then for all $y = (y_1, ..., y_n) \in \mathbb{R}^n$ such that $y_n > 0$, one has $\min_{u\in S^{n-1}} |P \cap (\mu y + \mathbb{R}u)| \underset{\mu\to 0}{\sim} |P \cap (\mu y + \mathbb{R}e_1)| \underset{\mu\to 0}{\sim} (8\mu y_n/k_1)^{\frac{1}{2}}$.*

Proof. As in the proof of Lemma 2 we may assume that $y \in P$, $\mu \leq 1$ and $u_n \neq 0$. We have $P \cap (\mu y + \mathbb{R}u) = \Big\{x \in \mathbb{R}^n\,;\ \exists\lambda \in \mathbb{R},\ x = \mu y + \lambda u \ \text{ and } x_n \geq \sum_{i=1}^{n-1} \frac{k_i}{2} x_i^2\Big\}$, so that $\{\lambda;\ \mu y + \lambda u \in P\} = [\lambda_1, \lambda_2]$, where $\lambda_1 < \lambda_2$ are the roots of the equation

$$\lambda^2\Big(\sum_{i=1}^{n-1} \frac{k_i}{2} u_i^2\Big) + \lambda\Big(-u_n + \mu\sum_{i=1}^{n-1} k_i y_i u_i\Big) - \mu\Big(y_n - \mu\sum_{i=1}^{n-1} \frac{k_i}{2} y_i^2\Big) \leq 0\,.$$

For $u \in S^{n-1}$, we define $f_\mu(u) = |P \cap (\mu y + \mathbb{R}u)|$. Since $|u| = 1$, we get $f_\mu(u) = |\lambda_2 - \lambda_1|$. We also define $H : \mathbb{R}^{n-1} \to \mathbb{R}$ by $H(x) = \sum_{i=1}^{n-1} \frac{k_i}{2} x_i^2$. We get

$$f_\mu(u) = H(u)^{-1}\Big(\Big(-u_n + \mu\sum_{i=1}^{n-1} k_i y_i u_i\Big)^2 + 4\mu H(u)(y_n - \mu H(y))\Big)^{\frac{1}{2}}.$$

Since $H(e_1) = k_1/2$, we have $\min_{u\in S^{n-1}} f_\mu(u) \leq f_\mu(e_1) \underset{\mu\to 0}{\sim} (8\mu y_n/k_1)^{\frac{1}{2}}$. On the other hand, $f_\mu(u) \geq \big(4\mu H(u)^{-1}(y_n - \mu H(y))\big)^{\frac{1}{2}}$. Since $|u| = 1$, we have $H(u) \leq k_1/2$; hence we get $\min_{u\in S^{n-1}} f_\mu(u) \geq (8\mu/k_1)^{\frac{1}{2}}(y_n - \mu H(y))^{\frac{1}{2}} \underset{\mu\to 0}{\sim} (8\mu y_n/k_1)^{\frac{1}{2}}$. □

Lemma 5. *Let $K \in \mathcal{K}^n$ with C^2 boundary and positive curvature. For all $x \in \partial K$,*

$$\lim_{t\to 0} \frac{1}{n}\langle x, N(x)\rangle \frac{1}{t^2}\left(1 - \rho_{K_1(t)}(x)^n\right) = \frac{1}{8}k_1(x).$$

Proof. 1) As in Lemma 3, since $\lim_{t\to 0}\lambda_t = 1$, using (1) there exists $t_1 > 0$ such that $s(t) := (1-\lambda_t)\langle x, N(x)\rangle \leq s_1$, for all $t \leq t_1$. Hence

$$t = \min_{u\in S^{n-1}} |K \cap (\lambda_t x + \mathbb{R}u)| \leq |P_{-\varepsilon} \cap (s(t)e_n + \mathbb{R}e_1)| = 2\Big(\frac{2s(t)}{(1-\varepsilon)k_1}\Big)^{\frac{1}{2}}.$$

Thus for $t \leq t_1$, one has $\dfrac{(1-\lambda_t(x))\langle x, N(x)\rangle}{t^2} = \dfrac{s(t)}{t^2} \geq (1-\varepsilon)\dfrac{1}{8}k_1(x)$.

2) For the reverse inequality, as in the proof of Lemma 3, we work with $S_x^{-1}(K)$ instead of K. Recall that $y = S_x^{-1}(0)$ and $\mu_t = 1 - \lambda_t$. For $t \leq t_1$ we get

$$t = \min_{u\in S^{n-1}} |K \cap (\lambda_t x + \mathbb{R}u)| \geq \min_{u\in S^{n-1}} |P_\varepsilon \cap H_{s_1}^+ \cap (\mu_t y + \mathbb{R}u)|.$$

Let $C_t = \{u \in S^{n-1};\ P_\varepsilon \cap H_{s_1}^- \cap (\mu_t y + \mathbb{R}u) \neq \emptyset\}$. It is clear that for $t \leq t_1/2$, we have $\min_{u \in C_t} |P_\varepsilon \cap H_{s_1}^+ \cap (\mu_t y + \mathbb{R}u)| \geq s_1/2$. If we apply Lemma 4 to P_ε, we see that

$$\min_{u \in S^{n-1}} |P_\varepsilon \cap H_{s_1}^+ \cap (\mu_t y + \mathbb{R}u)| \underset{t \to 0}{\sim} \min_{u \in S^{n-1}} |P_\varepsilon \cap (\mu_t y + \mathbb{R}u)| \underset{t \to 0}{\sim} \left(\frac{8\mu_t y_n}{k_1}\right)^{\frac{1}{2}}.$$

Thus there exists $t_2 > 0$ such that $t \geq (1+\varepsilon)^{-1}(8\mu_t y_n/k_1)^{\frac{1}{2}}$, for all $t \leq t_2$. We get $\dfrac{(1-\lambda_t)\langle x, N(x)\rangle}{t^2} \leq (1+\varepsilon)^2 k_1/8$ and we conclude since $\frac{1}{n}(1-\lambda_t^n) \underset{t\to 0}{\sim} 1-\lambda_t$. □

Proof of Theorems 3 and 4. Because of formula (2), the proof of Theorem 3 (respectively Theorem 4) is the immediate consequence of Lemmas 1 and 3 (resp. Lemmas 1 and 5) and the Lebesgue's theorem on dominated convergence. □

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