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GROWTH OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We consider the differential equation $f'' + A(z)f' + B(z)f = 0$, where $A(z)$ and $B(z)$ are entire functions. Provided $\rho(B) < \rho(A)$ and $T(r, A) \sim \log M(r, A)$ as $r \to \infty$ outside a set of finite logarithmic measure, we prove that all nonconstant solutions f of this equation are of infinite order.

1. Introduction

We consider the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

where $A(z), B(z) \not\equiv 0$ are entire functions and neither of them are polynomials. It is well known that each solution f of the equation (1.1) is an entire function, and if f_1 and f_2 are any two linearly independent solutions of (1.1), then at least one of f_1, f_2 must be of infinite order (see [10]).

In 1988, Gundersen [3] proved the following

Theorem A. *Let $A(z)$ and $B(z)$ be entire functions, where*
(i) $\rho(A) < \rho(B)$, *or*
(ii) *$A(z)$ is a polynomial and $B(z)$ is transcendental.*

Then every nonconstant solution f of (1.1) has infinite order.

In the same paper, Gundersen also proved

Theorem B. *If f is a nonconstant solution of (1.1), where*
(i) $\rho(B) < \rho(A) < \frac{1}{2}$, *or*
(ii) *$A(z)$ is transcendental with $\rho(A) = 0$ and $B(z)$ is a polynomial,*

then $\rho(f) = \infty$.

More recently, Hellerstein, Miles and Rossi [8] proved the following

Theorem C. *If $A(z)$ and $B(z)$ are entire functions with $\rho(B) < \rho(A) \le \frac{1}{2}$, then any nonconstant solution of (1.1) has infinite order.*

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In the proofs of Theorem B and Theorem C, the main tool is the classical $\cos \rho\pi$ theorem, which guarantees that the minimum modulus of $A(z)$ on a sequence of circles $|z| = R_n$, $n = 1, 2, \dots$, may be suitably large, if $\rho(A) < \frac{1}{2}$. When $\rho(A) = \frac{1}{2}$, the proof is more complicated. We know that if an entire function has a finite deficient value, then its order is $> \frac{1}{2}$ (see [1]). Thus, a natural question is: If $\rho(B) < \rho(A)$ and $A(z)$ has no finite deficient values, does every nonconstant solution of (1.1) have infinite order? This problem remains open here. In fact, we prove the same conclusion under a related condition on $A(z)$. Namely, a slight modification of the argument due to Hellerstein, Miles and Rossi in [8] implies

Theorem. *If $A(z)$ and $B(z)$ are entire functions with $\rho(B) < \rho(A) < \infty$ and $T(r, A) \sim \log M(r, A)$ as $r \to \infty$ outside a set G of finite logarithmic measure, then any nonconstant solution of* (1.1) *has infinite order.*

The notation $T(r, A) \sim \log M(r, A)$ of course means that

$$\lim_{r\to\infty} \frac{T(r, A)}{\log M(r, A)} = 1$$

holds outside of G. An entire function $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$ is said to have Fejér gaps (see [6] or [9]) if

$$\sum_{n=1}^{\infty} {\lambda_n}^{-1} < \infty.$$

If an entire function has Fejér gaps, then Murai [9] proved that $f(z)$ can have no finite deficient values. In the course of his proof, Murai showed that

$$T(r, f) \sim \log M(r, f)$$

holds as $r \to \infty$ outside a set of finite logarithmic measure. Therefore, we obtain

Corollary 1. *If $A(z)$ and $B(z)$ are entire functions with $\rho(B) < \rho(A) < \infty$ and $A(z)$ has Fejér gaps, then any nonconstant solution of* (1.1) *has infinite order.*

If $\rho(A) = \rho(B)$, the conclusion of our theorem is false in general. For example, if $P(z)$ is a polynomial, then $f(z) = e^{P(z)}$ solves (1.1) for arbitrary $A(z)$ with $B = -P'' - (P')^2 - A(z)P'$. If $\rho(B) < \rho(A)$, but $T(r, A) \sim \log M(r, A)$ does not hold as $r \to \infty$ outside a set of finite logarithmic measure, then the conclusion of our theorem also fails in general. For example, if $A(z) = e^z$, then $\log M(r, A) = r$ and $T(r, A) = r/\pi$, so that

$$\frac{T(r, A)}{\log M(r, A)} = \frac{1}{\pi}.$$

Now, the equation $f'' + e^z f' - f = 0$ is of the form (1.1), with a solution $f(z) = e^{-z} - 1$ of finite order.

2. Some known results and lemmas

To prove the theorem, we first establish a series of lemmas. To this end, we recall some notions and results. For $E \subset [1, \infty)$, we denote the complement of E in $[1, \infty)$ by CE, i.e. $CE = [1, \infty) \setminus E$. We define the linear measure of E by

$m(E) = \int_1^\infty \chi_E(t)\,dt$ where χ_E is the characteristic function of E, and the logarithmic measure of E by

$$m_l(E) = \int_1^\infty \frac{\chi_E(t)}{t}\,dt.$$

The upper and lower logarithmic density of E are defined by

$$\overline{\log}\,\overline{\text{dens}}\,E = \overline{\lim_{r\to\infty}} \frac{m_l(E\cap[1,r])}{\log r}$$

and

$$\underline{\log}\,\underline{\text{dens}}\,E = \underline{\lim_{r\to\infty}} \frac{m_l(E\cap[1,r])}{\log r},$$

respectively. We assume familiarity with the notations and fundamental results of the Nevanlinna theory (see [5]). In particular (see [5], p. 13),

$$|T(r,f) - T_0(r,f)| = O(1), \qquad r\to\infty,$$

where

$$T_0(r,f) = \int_0^r \frac{A(t,f)}{t}\,dt$$

is the Ahlfors–Shimizu characteristic, and $A(t,f)$ is the average number of solutions of $f(z) = a$ in $|z| \le t$ as a varies over the Riemann sphere. We will denote the order and lower order of f by $\rho(f)$ and $\mu(f)$.

Lemma 1 ([8, p. 697])**.** *For a nonconstant entire function f, let*

$$\varphi(r) = \frac{1}{2\pi}\int_0^{2\pi} n(r, e^{i\alpha}, f)\,d\alpha$$

be the mean covering number of the unit circle under the map f restricted to $\{\, z : |z| \le r \,\}$. Then there exists a set E_1 with $m_l(E_1) < \infty$ such that

$$\lim_{\substack{r\to\infty\\ r\notin E_1}} \frac{\varphi(r)}{A(r,f)} = 1.$$

Lemma 2 ([2, p. 90])**.** *Let $f(z)$ be a meromorphic function of finite order ρ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E_2(\varepsilon)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E_2(\varepsilon) \cup [0,1]$ and for all integers $k > j \ge 0$ in a given finite set of pairs (k,j), we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Lemma 3. *Suppose that $f(z)$ is an entire function of order $\rho < \infty$. Let $n(r) = n(r, \frac{1}{f})$ and suppose $n(1) \ge 1$. For $K > 1$, $h > 0$, let*

$$E_3(K,h) = \{\, r \ge e : n(r) > K n(r/e^h) \,\}.$$

Then

$$\underline{\log}\,\underline{\text{dens}}\,CE_3(K,h) \ge 1 - \frac{h\rho}{\log K}, \tag{2.1}$$

and consequently,

$$\overline{\log}\,\overline{\text{dens}}\,E_3(K,h) \le h\rho/\log K. \tag{2.2}$$

Proof. Fixing $\varepsilon > 0$, there exists $r_0 > e$ such that

$$1 \le n(r) < r^{\rho+\varepsilon} \tag{2.3}$$

for all $r > r_0$. Consider now the inequality

$$n(t) \le Kn(t/e^h). \tag{2.4}$$

First of all, if there exists $r_1 > r_0$ such that for all $r > r_1$, the inequality (2.4) holds for all $t \in [r_1, r]$, then (2.1) is trivial. So, we may assume, without loss of generality, that for any given $r > r_0$, there exists a value $t \in [r_0, r]$ such that (2.4) does not hold. Let

$$t_1' = \sup\{\, t \in [r_0, r] : n(t) > Kn(te^{-h})\}.$$

If $n(t_1') \ge Kn(t_1' e^{-h})$, we define $t_1 = t_1'$. Otherwise, we choose t_1 such that $0 < t_1' - t_1 < \varepsilon$ and

$$n(t_1) > Kn(t_1 e^{-h}).$$

Suppose now, inductively, that t_{m-1} has been defined similarly as for t_1 above. Then let

$$t_m' = \sup\{\, t \in [r_0, t_{m-1}e^{-h}] : n(t) > Kn(te^{-h})\,\}.$$

If $n(t_m') \ge Kn(t_m' e^{-h})$, we define $t_m = t_m'$. Otherwise, we choose t_m such that $0 < t_m' - t_m < \frac{\varepsilon}{2^{m-1}}$ and

$$n(t_m) > Kn(t_m e^{-h}).$$

This inductive process now terminates either if (2.4) is satisfied for all $t \in [r_0, t_{m-1}e^{-h}]$ or if $t_m e^{-h} \le r_0 < t_m$. From the above inequalities, we have

$$\begin{aligned} r^{\rho+\varepsilon} &\ge n(r) \ge n(t_1') \ge n(t_1) \ge Kn(t_1 e^{-h}) \ge Kn(t_2) \\ &\ge K^2 n(t_2 e^{-h}) \ge \cdots \ge K^{m-1} n(t_m) \ge K^{m-1} n(r_0) \ge K^{m-1}. \end{aligned}$$

Hence

$$(m-1)h < \frac{(\rho+\varepsilon)h}{\log K} \log r,$$

i.e.

$$mh < \frac{(\rho+\varepsilon)h}{\log K} \log r + h.$$

Therefore,

$$\begin{aligned} \int_{CE_3(K,h)\cap[r_0,r]} \frac{dt}{t} &\ge \int_{r_0}^{r} \frac{dt}{t} - mh - \sum_{j=1}^{m} \int_{t_j}^{t_j'} \frac{dt}{t} \\ &\ge \log\frac{r}{r_0} - \frac{(\rho+\varepsilon)h}{\log K}\log r - h - \sum_{j=1}^{m} \log\frac{t_j'}{t_j} \\ &\ge \log\frac{r}{r_0} - \frac{(\rho+\varepsilon)h}{\log K}\log r - h - \sum_{j=1}^{m} \frac{\varepsilon}{2^{j-1} r_0} \\ &\ge \log\frac{r}{r_0} - \frac{(\rho+\varepsilon)h}{\log K}\log r - h - \frac{2\varepsilon}{r_0}. \end{aligned}$$

Hence,

$$\liminf_{r\to\infty} \frac{1}{\log r} \int_{CE_3(K,h)\cap[r_0,r]} \frac{dt}{t} \geq 1 - \frac{(\rho+\varepsilon)h}{\log K}.$$

Since $\varepsilon > 0$ is arbitrary, (2.1) and so (2.2) follow by $\varepsilon \to 0$.

Lemma 4. *Let $T(r) > 1$ be a nonconstant increasing function of finite order ρ in $r \in (1,\infty)$, i.e.*

$$\limsup_{r\to\infty} \frac{\log T(r)}{\log r} = \rho < \infty.$$

For any ρ_3 such that $0 \leq \rho_3 < \rho$, if $\rho > 0$, and $\rho_3 = 0$, if $\rho = 0$, define

$$E_4(\rho_3) = \{\, r \geq 1 : r^{\rho_3} < T(r) \,\}.$$

Then

$$\delta := \overline{\log}\,\overline{\text{dens}}\, E_4(\rho_3) > 0. \tag{2.5}$$

Proof. If $\rho_3 = 0$, then (2.5) holds trivially. Assume now that $0 < \rho_3 < \rho$. If $\overline{\log}\,\overline{\text{dens}}\, E_4(\rho_3) = 0$, then for any given real numbers $\alpha > 1$, and η such that $0 < \eta < \frac{\alpha-1}{\alpha+1}$, there exists $r_0 > 1$ such that

$$\frac{1}{\log r^\alpha} \int_{E_4(\rho_3)\cap[1,r^\alpha]} \frac{dt}{t} < \eta$$

for all $r > r_0$ by the definition of upper logarithmic density. Therefore,

$$\int_{E_4(\rho_3)\cap[r,r^\alpha]} \frac{dt}{t} < \alpha\eta \log r \tag{2.6}$$

for all $r > r_0$. By (2.6),

$$\begin{aligned}(\alpha-1)\log r = \int_r^{r^\alpha} \frac{dt}{t} &= \int_{E_4(\rho_3)\cap[r,r^\alpha]} \frac{dt}{t} + \int_{CE_4(\rho_3)\cap[r,r^\alpha]} \frac{dt}{t} \\ &< \alpha\eta\log r + \int_{CE_4(\rho_3)\cap[r,r^\alpha]} \frac{dt}{t}.\end{aligned}$$

Since $\eta < \frac{\alpha-1}{\alpha+1}$, a simple computation results in

$$\eta \log r < \int_{CE_4(\rho_3)\cap[r,r^\alpha]} \frac{dt}{t}. \tag{2.7}$$

By (2.7), there exists $t \in CE_4(\rho_3) \cap [r, r^\alpha]$ such that

$$T(r) \leq T(t) \leq t^{\rho_3} \leq r^{\alpha\rho_3}$$

for all $r > r_0$. This implies that $\rho \leq \alpha\rho_3$. Since $\alpha > 1$ is arbitrary, we obtain $\rho \leq \rho_3$ by $\alpha \to 1$, a contradiction.

Lemma 5. *Suppose that $\varphi_j(t)$, $j = 1, 2$, are nondecreasing, nonnegative functions in $[1,\infty)$ such that for a real constant C,*

$$\int_1^r \frac{\varphi_1(t)}{t^2}\,dt \leq \int_1^r \frac{\varphi_2(t)}{t^2}\,dt + C$$

for all $r > 1$. For any given constant $K > 1$, define

$$E_5(K) = \{\, t > 1 : \varphi_1(t) \geq K e \varphi_2(t) \,\}.$$

Then

$$\overline{\log}\,\overline{\text{dens}}\,E_5(K) \le \frac{1}{K}.$$

Proof. This is an immediate consequence of [7], Theorem 4, and [6], Theorem 3.

Lemma 6. *Let f be a nonconstant entire function, and $K > 1$ a given constant, and define*

$$E_6(K) = \{\, t \ge 1 : T_0(t,f) > eKA(t,f)\,\}.$$

Then

$$\overline{\log}\,\overline{\text{dens}}\,E_6(K) \le \frac{1}{K}.$$

Proof. By partial integration,

$$\int_1^r \frac{T_0(t,f)}{t^2}\,dt = \int_1^r \frac{A(t,f)}{t^2}\,dt + T_0(1,f) - \frac{T_0(r,f)}{r} \le \int_1^r \frac{A(t,f)}{t^2}\,dt + T_0(1,f).$$

By Lemma 5,

$$\overline{\log}\,\overline{\text{dens}}\,E_6(K) \le \frac{1}{K}.$$

Making use of the notations of the preceding lemmas, we easily obtain

Lemma 7. *Suppose that $f(z)$ is a nonconstant entire function of finite order ρ, $0 < \rho < \infty$, satisfying*

$$\lim_{\substack{r\to\infty \\ r\notin G}} \frac{T(r,f)}{\log M(r,f)} = 1, \tag{2.8}$$

where G is a set of finite logarithmic measure. Then, for any given number $K > \max(\frac{16}{7\delta}, e^{16\rho/\delta})$, there exists a sequence $\{R_n\}$, $R_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{T(R_n,f)}{\log M(R_n,f)} = 1, \tag{2.9}$$

$$n(R_n) \le Kn(R_n/e), \tag{2.10}$$

$$\lim_{n\to\infty} \frac{\varphi(R_n)}{A(R_n,f)} = 1, \tag{2.11}$$

$$T_0(R_n,f) \le eKA(R_n,f), \tag{2.12}$$

and

$$R_n^{\rho_3} < T(R_n,f) \le \log M(R_n,f). \tag{2.13}$$

Proof. Let

$$F := E_4(\rho_3) \setminus (E_1 \cup E_2(1) \cup E_3(K,1) \cup E_6(K) \cup G).$$

Then

$$E_4(\rho_3) \subset F \cup E_1 \cup E_2(1) \cup E_3(K,1) \cup E_6(K) \cup G,$$

and so

$$\delta < \overline{\log}\,\overline{\text{dens}}\,F + \delta/16 + 7\delta/16.$$

Thus

$$\frac{\delta}{2} < \overline{\log}\,\overline{\text{dens}}\, F.$$

Therefore, there exists a sequence $\{R_n\}$, $R_n \in F$, tending to infinity such that (2.9)–(2.13) hold.

Lemma 8. *Suppose that $f(z)$ is a meromorphic function of finite order ρ and a sequence $\{R_n\}$ has been defined as in Lemma 7. Then, for any given number h, $0 < h < \infty$, we have*

$$T(R_n e^h, f) < e^{h\rho} T(R_n, f)\big(1 + \circ(1)\big)$$

as $n \to \infty$.

Proof. See [11], pp. 40–43.

Lemma 9. *Suppose that $f(z)$ is a nonconstant entire function satisfying the conditions of Lemma 7. Let $\{R_n\}$ be defined as in Lemma 7, let $0 < \rho_2 < \rho_3 < \rho$, and define*

$$E(R_n) := \{\, \theta \in [0, 2\pi) : \log |f(R_n e^{i\theta})| \leq R_n^{\rho_2} \,\}.$$

Then, for any given constant $\varepsilon > 0$, there exists n_0 such that

$$m\big(E(R_n)\big) \leq \varepsilon \tag{2.14}$$

for all $n \geq n_0$.

Proof. Since $\delta(\infty, f) = 1$, it follows from (2.9) that

$$\begin{aligned}\Big(1 - \frac{\varepsilon}{4\pi}\Big) \log M(R_n, f) &\leq \frac{1}{2\pi} \int_0^{2\pi} \overset{+}{\log} |f(R_n e^{i\theta})| d\theta \\ &\leq R_n^{\rho_2} + \frac{m\big(CE(R_n)\big)}{2\pi} \log M(R_n, f),\end{aligned}$$

where $CE(R_n) = [0, 2\pi) \setminus E(R_n)$. It now follows that

$$2\pi \Big(1 - \frac{\varepsilon}{4\pi}\Big) - \frac{2\pi R_n^{\rho_2}}{\log M(R_n, f)} < m\big(CE(R_n)\big). \tag{2.15}$$

Since $\rho_2 < \rho_3$, $R_n \in E_4(\rho_3)$ and $T(R_n, f) \leq \log M(R_n, f)$, from (2.15) we obtain

$$2\pi \Big(1 - \frac{\varepsilon}{2\pi}\Big) < m\big(CE(R_n)\big), \tag{2.16}$$

provided that $n \geq n_0$ and n_0 is sufficiently large. Therefore,

$$m\big(E(R_n)\big) \leq \varepsilon.$$

3. Proof of the Theorem

Assume that f is a nonconstant entire solution of finite order $\rho(f)$ of (1.1). If $f(z)$ has finitely many zeros only, then $f(z) = P(z)e^{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials. Substituting this into (1.1) gives

$$P'' + 2P'Q' + PQ'' + (Q')^2 P + A(z)(P' + Q'P) + B(z)P = 0. \tag{3.1}$$

Since $\rho(B) < \rho(A)$, it follows from (3.1) that $P' + Q'P \equiv 0$. Thus $Q' \equiv 0$ and $P' \equiv 0$. Then f is a constant, a contradiction. Therefore, we may assume that

$f(z)$ has infinitely many zeros $\{a_\nu\}$. Now let $\{R_n\}$ be the sequence determined by Lemma 7 with respect to $A(z)$, and proceed to estimate

$$\frac{1}{2\pi}\int_{E(R_n)}\left(ReR_ne^{i\theta}\frac{f'(R_ne^{i\theta})}{f(R_ne^{i\theta})}\right)^+ d\theta.$$

To this end, we apply the differential Poisson–Jensen formula (see [5], p. 22), to obtain

(3.2)

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\varphi})|\frac{2zRe^{i\varphi}}{(Re^{i\varphi}-z)^2}d\varphi+\sum_{|a_\nu|\le r/e}\left(\frac{z}{z-a_\nu}+\frac{\bar{a}_\nu z}{R^2-\bar{a}_\nu z}\right)\\ &\quad+\sum_{r/e<|a_\nu|\le r}\left(\frac{z}{z-a_\nu}+\frac{\bar{a}_\nu z}{R^2-\bar{a}_\nu z}\right)+\sum_{r<|a_\nu|<R}\left(\frac{z}{z-a_\nu}+\frac{\bar{a}_\nu z}{R^2-\bar{a}_\nu z}\right)\\ &:= f_1+f_2+f_3+f_4,\end{aligned}$$

where $|z|=r<R$; see also [8], pp. 696–697.

To apply (3.2), we choose $R=\frac{R_ne^h}{2}$, $h=1+\log 4$. By the first main theorem, Lemma 8 and (2.12), we first obtain

$$|f_1(R_ne^{i\theta})|\le 18e^{1+h\rho}T_0(R_n,f)\le 18e^{2+h\rho}KA(R_n,f),$$

so that

$$\frac{1}{2\pi}\int_{E(R_n)}|f_1(R_ne^{i\theta})|d\theta\le\frac{9e^{2+h\rho}}{\pi}KA(R_n,f)m\big(E(R_n)\big). \tag{3.3}$$

By the formula (3.5) in [8], we have

$$\frac{1}{2\pi}\int_{E(R_n)}|Ref_2(R_ne^{i\theta})|d\theta\le\frac{2}{\pi}n(R_n/e)m\big(E(R_n)\big) \tag{3.4}$$

and by [8], (3.3) and (3.4),

$$\frac{1}{2\pi}\int_{E(R_n)}|Ref_3(R_ne^{i\theta})|d\theta\le n(R_n)-n(R_n/e). \tag{3.5}$$

From (3.7) in [8], Lemma 8 and (2.12), we deduce

$$\begin{aligned}\frac{1}{2\pi}\int_{E(R_n)}\big(Ref_4(R_ne^{i\theta})\big)^+d\theta &\le\frac{1}{2\pi}\left(n\left(\frac{R_ne^h}{2}\right)-n(R_n)\right)m\big(E(R_n)\big)\\ &\le\frac{1}{2\pi}n\left(\frac{R_ne^h}{2}\right)m\big(E(R_n)\big)\le\frac{1}{2\pi\log 2}N(R_ne^h)m(E(R_n))\\ &\le\frac{1}{2\pi\log 2}T(R_ne^h,f)m(E(R_n))\le\frac{e^{h\rho}}{2\pi\log 2}T(R_n,f)m\big(E(R_n)\big)\\ &\le\frac{e^{h\rho}}{\pi\log 2}T_0(R_n,f)m\big(E(R_n)\big)\le\frac{e^{1+h\rho}}{\pi\log 2}KA(R_n,f)m\big(E(R_n)\big).\end{aligned} \tag{3.6}$$

Combining now (3.2)–(3.6), we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{E(R_n)}\left(ReR_ne^{i\theta}\frac{f'(R_ne^{i\theta})}{f(R_ne^{i\theta})}\right)^+d\theta &\le\frac{10e^{2+h\rho}K}{\pi}A(R_n,f)m\big(E(R_n)\big)\\ &\quad+\frac{2}{\pi}n(R_n)m\big(E(R_n)\big)+n(R_n)-n(R_n/e).\end{aligned} \tag{3.7}$$

As in [8], p. 698, we apply the argument principle to deduce

$$\frac{1}{2\pi}\int_0^{2\pi} n(R_n, e^{i\alpha}, f)\, d\alpha = \frac{1}{2\pi}\int_{B_{R_n}} ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\, d\theta,$$

where

$$B_{R_n} = \{\, \theta \in [0, 2\pi) : |f(R_n e^{i\theta})| > 1\,\}.$$

Moreover, by (2.11) and Lemma 1 we conclude

$$\begin{aligned} A(R_n, f) &\le \frac{1+o(1)}{2\pi}\int_0^{2\pi}\left(ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right)^+ d\theta \\ &= \frac{1+o(1)}{2\pi}\int_{E(R_n)}\left(ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right)^+ d\theta \\ &\quad + \frac{1+o(1)}{2\pi}\int_{CE(R_n)}\left(ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right)^+ d\theta. \end{aligned} \tag{3.8}$$

We now choose ρ_1, ρ_2, ρ_3 such that $\rho(B) < \rho_1 < \rho_2 < \rho_3 < \rho(A)$. If $\theta \in CE(R_n)$, then from (1.1) and Lemma 2 applied to $f(z)$ and Lemma 9 to $A(z)$, we have

$$\left| R_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right| \le \frac{e^{R_n^{\rho_1}} + R_n^q}{e^{R_n^{\rho_2}}} \to 0, \tag{3.9}$$

as $n \to \infty$, where q is a constant. It follows from (3.8) and (3.9) that

$$A(R_n, f) \le \frac{2}{2\pi}\int_{E(R_n)}\left(ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right)^+ d\theta \tag{3.10}$$

as n is sufficiently large. By (3.7), (3.10) and (2.10) we deduce that

(3.11)

$$\begin{aligned} &\frac{1}{2\pi}\int_{E(R_n)}\left(ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right)^+ d\theta \\ &\quad\le \frac{10e^{2+h\rho}K}{\pi}A(R_n, f)m\big(E(R_n)\big) + \frac{2}{\pi}n(R_n)m\big(E(R_n)\big) + n(R_n) - \frac{1}{K}n(R_n) \\ &\quad\le \frac{10e^{2+h\rho}K}{\pi}m\big(E(R_n)\big)\frac{1}{\pi}\int_{E(R_n)}\left(ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right)^+ d\theta \\ &\qquad + \frac{2}{\pi}n(R_n)m\big(E(R_n)\big) + n(R_n) - \frac{1}{K}n(R_n). \end{aligned}$$

Now fix $0 < \varepsilon < \frac{\pi}{40e^{2+h\rho}K^2}$. By Lemma 9, there exists n_0 such that

$$m\big(E(R_n)\big) < \varepsilon \tag{3.12}$$

whenever $n > n_0$. It follows from (3.11) and (3.12) that

$$\begin{aligned}\frac{1}{2\pi}\int_{E(R_n)}\left(ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right)^{+}d\theta \\ &\leq \frac{10e^{2+h\rho}K\varepsilon}{\pi}\frac{1}{\pi}\int_{E(R_n)}\left(ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right)^{+}d\theta \\ &\quad+\frac{2\varepsilon}{\pi}n(R_n)+n(R_n)-\frac{1}{K}n(R_n) \\ &< \frac{1}{2K}\frac{1}{2\pi}\int_{E(R_n)}\left(ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right)^{+}d\theta \\ &\quad+\frac{1}{20e^{2+h\rho}K^2}n(R_n)+n(R_n)-\frac{1}{K}n(R_n).\end{aligned}$$

Therefore,

$$\begin{aligned}\left(1-\frac{1}{2K}\right)\frac{1}{2\pi}\int_{E(R_n)}\left(ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}\right)^{+}d\theta \\ < \frac{1}{20e^{2+h\rho}K^2}n(R_n)+n(R_n)-\frac{1}{K}n(R_n).\end{aligned} \tag{3.13}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}ReR_n e^{i\theta}\frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})}d\theta = n(R_n),$$

it follows from (3.9) and (3.13) that

$$\left(1-\frac{1+o(1)}{2K}\right)n(R_n) < \frac{1}{20e^{2+h\rho}K^2}n(R_n)+n(R_n)-\frac{1}{K}n(R_n).$$

Hence

$$\frac{1+o(1)}{2} < \frac{1}{20e^{2+h\rho}K}.$$

Taking K as in Lemma 7 large enough, we get a contradiction. The proof of the Theorem is completed.

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