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ON THE PARAMETRIZATION OF SELF-SIMILAR AND OTHER FRACTAL SETS

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Abstract. We prove for many self-similar, and some more general, sets $E \subset \mathbb{R}^n$ that if s is the Hausdorff dimension of E and $f : \mathbb{R}^m \to \mathbb{R}^n$ is Hölder continuous with exponent m/s, then the s-dimensional Hausdorff measure of $E \cap f(\mathbb{R}^m)$ is 0.

1. Introduction

In this paper we shall study the following question, and some more general forms of it: which s-dimensional self-similar subsets of $\mathbb{R}^n$ can be parametrized by (m/s)-Hölder continuous maps from $\mathbb{R}^m$? More precisely, let $E \subset \mathbb{R}^n$ be a compact self-similar set. This means that there are contracting similarities $S_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, ..., N$, $N \geq 2$, such that

$$E = \bigcup_{i=1}^{N} S_i(E).$$

Our question is: when does there exist $f : A \longrightarrow \mathbb{R}^n$, $A \subset \mathbb{R}^m$, such that

$$| f(x) - f(y) | \leq c \mid x - y |^{m/s} \quad \text{for } x, y \in A, \tag{1}$$

and $f(A) = E$? We shall call maps f satisfying (1) (m/s)-Hölder maps.

Obviously, when the Hausdorff dimension, $\dim E$, of E is s, we can never replace m/s with a larger exponent. On the other hand, if we allow a smaller exponent, such parametrizations can be easily found for many sets E; see Theorem 3.4.

It was shown in [MM2] that if the similarities S_i can be chosen so that the different parts $S_i(E)$ are disjoint, then the above parametrization is impossible. In fact, then the s-dimensional Hausdorff measure

$$H^s(E \cap f(A)) = 0 \tag{2}$$

for all such (m/s)-Hölder maps f. In section 3 we give simpler proofs of this result in specific cases.

In Theorem 2.1 we consider a general class of disconnected but not necessarily totally disconnected sets for which (2) holds for all (m/s)-Hölder maps as above.

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This applies to many self-similar sets, a typical example being the cartesian product of a Cantor set in $\mathbb{R}$ with an interval; see Theorem 2.2.

Recently Remes has given in [R] rather general sufficient conditions for the existence of $(1/s)$-parametrizations from $\mathbb{R}$. For example, he has shown that any connected self-similar set satisfying the open set condition admits such a parametrization. Some of the sets he can parametrize are disconnected. The following question, which we formulate only in the most basic case $m = 1$, is left open:

Question 1.1. Let $E \subset \mathbb{R}^n$ be a self-similar set with uncountably many connected components, $s = \dim E > 1$, and suppose that E satisfies the open set condition. Is $H^s(E \cap f(\mathbb{R})) = 0$ for every $(1/s)$-Hölder-map $f : \mathbb{R} \to \mathbb{R}^n$?

We shall introduce the basic notation we will use. We denote by $B(x, r)$ the closed ball with center x and radius r, by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$, by $\alpha(n) = \mathcal{L}^n(B(0, 1))$ the volume of the unit ball, and by H^s the s-dimensional Hausdorff measure. For $x \in \mathbb{R}^n$, $A, B \subset \mathbb{R}^n$, $d(x, A)$, $d(A, B)$ and $d(A)$ will be the distance from x to A, the distance between A and B, and the diameter of A, respectively. By χ_A we denote the characteristic function of A.

2. Hölder-maps and product-like sets

In this section we consider sets which have a local structure similar to that of the product of a Cantor set and an interval. We first prove a general result which states that it is impossible to cover a set of that type with a Hölder-image of $\mathbb{R}^m$ with Hölder exponent $m/\dim E$.

Theorem 2.1. *Suppose that E is a Borel set in $\mathbb{R}^n$, μ is a Borel measure in $\mathbb{R}^n$ and s is a positive number such that $\mu(E) > 0$ and*

$$\mu(B(x, r)) \le r^s \text{ for } x \in \mathbb{R}^n \text{ and } r > 0. \tag{3}$$

Suppose further that there is a sequence $\delta_k \searrow 0$ and that for every $\eta > 0$ there is $M_\eta > 0$ such that the following holds:

For every $k = 1, 2, ...$ there are Borel sets $E_{k,1}, ..., E_{k,m_k}$ such that

$$E = \bigcup_{i=1}^{m_k} E_{k,i}, \tag{4}$$

$$d(E_{k,i}, E_{k,j}) > \delta_k \text{ for } i \ne j, \text{ and} \tag{5}$$

$$\mu(E_{k,i} \cap B(x, r)) < \eta\mu(B(x, r)) \text{ for all } x \in E \text{ and } r > M_\eta\delta_k. \tag{6}$$

Then for any (m/s)-Hölder-map $f : A \to \mathbb{R}^n, A \subset \mathbb{R}^m$, we have

$$\mu(E \cap f(A)) = 0.$$

Proof. Suppose, contrary to the assertion, that there exists an (m/s)-Hölder-map $f : A \to \mathbb{R}^n$, $A \subset \mathbb{R}^m$, with $\mu(E \cap f(A)) > 0$. Since Hölder-maps always extend (see, e.g., [S, VI 2.2, Theorem 3] or [MM2, Lemma 3.1]), we may assume that $A = \mathbb{R}^m$. Applying a homothety we may also assume that

$$|f(x) - f(y)| \le |x - y|^{m/s} \text{ for } x, y \in \mathbb{R}^m. \tag{7}$$

There is some open ball B for which $\mu(E \cap f(B)) > 0$. Let z be a μ-density point of $E \cap f(B)$, that is,

$$\lim_{r \to 0} \frac{\mu(E \cap f(B) \cap B(z,r))}{\mu(B(z,r))} = 1.$$

(See for example [M, 2.14] for the fact that μ almost all points of $E \cap f(B)$ are μ-density points.) Choose $r_0 > 0$ such that

$$\mu(B(z,r_0) \backslash (E \cap f(B))) < \frac{1}{4}\mu(B(z,r_0)). \tag{8}$$

Set

$$F = E \cap f(B) \cap B(z,r_0) \text{ and } C = B \cap f^{-1}(F). \tag{9}$$

Let $P(m)$ be the constant in Besicovitch's covering theorem as in [M, Theorem 2.7(1)], and set $C(\delta) = \{x \in \mathbb{R}^m : d(x,C) \le \delta\}$ for $\delta > 0$. Let η be such that

$$0 < \eta < 8^{-m}\alpha(m)P(m)^{-1}\mathcal{L}^m(C(1))^{-1}\mu(B(z,r_0)) \tag{10}$$

and let $M = M_\eta$ be as in the assumptions of the theorem. We have

$$\lim_{\delta \to 0} \mathcal{L}^m(C(\delta) \backslash C) = 0$$

and we can choose k_0 such that $\delta_{k_0} < 1$ and

$$\mathcal{L}^m(C(\delta_{k_0}^{s/m}) \backslash C) < P(m)^{-1}\alpha(m)6^{-m}M^{-s}\frac{1}{2}\mu(B(z,r_0)). \tag{11}$$

Set

$$\delta = \delta_{k_0}, F_i = E_{k_0,i} \cap F \text{ and } C_i = B \cap f^{-1}(F_i). \tag{12}$$

For each $x \in C$ let $r(x)$ be the largest radius such that

$$B(x, r(x)) \subset C(\delta^{s/m}/2).$$

Then $r(x) \ge \delta^{s/m}/2$ and there is $x' \in \mathbb{R}^m$ such that

$$B(x', \delta^{s/m}/6) \subset B(x, r(x)) \backslash C.$$

Applying Besicovitch's covering theorem [M, 2.7], we find closed balls $B_i = B(x_i, r(x_i)), x_i \in C$, and $B_i' = B(x_i', \delta^{s/m}/6)$ such that

$$C \subset \bigcup_i B_i \subset C(\delta^{s/m}/2), \tag{13}$$

$$\sum_i \chi_{B_i} \le P(m), \tag{14}$$

$$B_i' \subset B_i \backslash C \text{ and } d(B_i') = \delta^{s/m}/3. \tag{15}$$

Since $C = \bigcup_j C_j$, by (9), (12) and (4), we have for each i , $x_i \in C_{j_i}$ for some j_i. By (5) and (12), $d(F_j, F_k) > \delta$ for $j \ne k$. Hence (7) and (12) yield $d(C_j, C_k) > \delta^{s/m}$ for $j \ne k$, and so $d(C_{j_i}, C \backslash C_{j_i}) > \delta^{s/m}$. Since $B_i \subset C(\delta^{s/m}/2)$ by (13), B_i cannot meet C_k for $k \ne j_i$. We conclude that there is a unique j_i such that

$$C_{j_i} \cap B_i \ne \emptyset \text{ and } f(C \cap B_i) \subset F_{j_i}. \tag{16}$$

Pick $y_i \in F_{j_i} \cap f(B_i)$ and set

$$D_i = B(y_i, d(f(B_i))) \supset f(B_i). \tag{17}$$

Then by (3), (7), (13), (14) and the fact that $\delta < 1$,

$$\begin{aligned}\sum_i \mu(D_i) &\le \sum_i d(f(B_i))^s \le \sum_i d(B_i)^m \\ &= 2^m\alpha(m)^{-1}\sum_i \mathcal{L}^m(B_i) = 2^m\alpha(m)^{-1}\int_{C(\delta^{s/m}/2)}\sum_i \chi_{B_i}d\mathcal{L}^m \\ &\le 2^m\alpha(m)^{-1}P(m)\mathcal{L}^m(C(1)). \qquad (18)\end{aligned}$$

Set

$$I = \{i : d(f(B_i)) > M\delta\}, \qquad (19)$$

$$J = \{i : d(f(B_i)) \le M\delta\}. \qquad (20)$$

Then, as $f(C \cap B_i) \subset F_{j_i} \cap D_i$ by (16) and (17), we have using (6), (19), (18) and (10),

$$\begin{aligned}\mu(\bigcup_{i\in I} f(C\cap B_i)) &\le \sum_{i\in I}\mu(f(C\cap B_i)) \\ &\le \sum_{i\in I}\mu(F_{j_i}\cap D_i) \le \eta\sum_{i\in I}\mu(D_i) \\ &\le \eta 2^m\alpha(m)^{-1}P(m)\mathcal{L}^m(C(1)) < (1/4)\mu(B(z,r_0)). \qquad (21)\end{aligned}$$

Since by (9) and (13),

$$E\cap f(B)\cap B(z,r_0) = F \subset \bigcup_{i\in I} f(C\cap B_i) \cup \bigcup_{i\in J} f(C\cap B_i),$$

(21) together with (8) yields

$$\begin{aligned}\mu(\bigcup_{i\in J} f(C\cap B_i)) &\ge \mu(B(z,r_0)) - \mu(\bigcup_{i\in I} f(C\cap B_i)) - \mu(B(z,r_0)\backslash(E\cap f(B))) \\ &\ge \frac{1}{2}\mu(B(z,r_0)).\end{aligned}$$

Recalling also (13), (15), (14), (20), (17) and (3) we get from this

$$\begin{aligned}\mathcal{L}^m(C(\delta^{s/m})\backslash C) &\ge \mathcal{L}^m(\bigcup_{i\in J} B_i') \ge \int P(m)^{-1}\sum_{i\in J}\chi_{B_i'}d\mathcal{L}^m \\ &= P(m)^{-1}(\text{card } J)\alpha(m)(\delta^{s/m}/6)^m \ge P(m)^{-1}\alpha(m)6^{-m}M^{-s}\sum_{i\in J} d(f(B_i))^s \\ &\ge P(m)^{-1}\alpha(m)6^{-m}M^{-s}\sum_{i\in J}\mu(D_i) \ge P(m)^{-1}\alpha(m)6^{-m}M^{-s}\sum_{i\in J}\mu(f(B_i)) \\ &\ge P(m)^{-1}\alpha(m)6^{-m}M^{-s}\frac{1}{2}\mu(B(z,r_0)).\end{aligned}$$

This contradicts (11) and proves the theorem.

Remarks. 1) By Frostman's lemma [M, Theorem 8.8] the existence of a measure μ satisfying $\mu(E) > 0$ and (3) is equivalent to $H^s(E) > 0$.

2) Theorem 2.1 applies if E is a self-similar set with disjoint parts, that is, there are contracting similarities $S_1, ..., S_N, N \ge 2$, such that

$$E = \bigcup_{i=1}^N S_i(E) \text{ and } S_i(E)\cap S_j(E) = \emptyset \text{ for } i \ne j.$$

More generally it applies under the assumptions of the following theorem. However, we do not know if any set E as in Question 1.1 has a structure to which Theorem 2.1 or the method of its proof applies.

Theorem 2.2. *Let $E' \subset \mathbb{R}^p$ be a self-similar set with disjoint parts, $t = \dim E'$ and $F \subset \mathbb{R}^{n-p}$ such that with some positive number c_1,*

$$r^{s-t}/c_1 \le H^{s-t}(F \cap B(y,r)) \le c_1 r^{s-t} \text{ for } y \in F \text{ and } 0 < r < 1. \tag{22}$$

Then for any (m/s)-Hölder map $f : A \to \mathbb{R}^n$, $A \subset \mathbb{R}^m$, we have

$$H^s((E' \times F) \cap f(A)) = 0.$$

Proof. We may assume $d(E) < 1$. Let us show that the conditions (3)-(6) in Theorem 2.1 hold for $E = E' \times F$. We take $\mu = (H^t \mid E') \times (H^{s-t} \mid F)$ where $H^u \mid A$ is the restriction of H^u to A. There is a positive number c_2 such that

$$r^t/c_2 \le H^t(E' \cap B(x,r)) \le c_2 r^t \text{ for } x \in E' \text{ and } 0 < r < 1; \tag{23}$$

see [H] or [F]. It follows that with $c_3 = 2c_1c_2$,

$$r^s/c_3 \le \mu(E \cap B(x,r)) \le c_3 r^s \text{ for } x \in E \text{ and } 0 < r < 1. \tag{24}$$

This gives (3) for $c_3^{-1}\mu$.

Let $E' = \bigcup_{i=1}^N S_i(E')$, where the sets $S_i(E')$ are disjoint. Let $r_1 \le ... \le r_N < 1$ be the similarity ratios of $S_1, ..., S_N$ and

$$\theta = \min\{d(S_i(E'), S_j(E')) : i \ne j\} > 0.$$

For each $k = 2, 3, ...$, let $E'_{k,i}, i = 1, ..., m_k$, be all the sets $S_{i_1} \circ ... \circ S_{i_l}(E')$ such that

$$r_{i_1}...r_{i_l} \le r_1^k < r_{i_1}...r_{i_{l-1}}.$$

Then

$$d(E'_{k,i}, E'_{k,j}) \ge r_1^k \theta \text{ for } i \ne j. \tag{25}$$

We take now $E_{k,i} = E'_{k,i} \times F$ and $\delta_k = r_1^k\theta$. Then (4) and (5) are satisfied. We show that (6) holds for $M_\eta = (c_1 c_3^{-1} \eta^{-1} H^t(E'))^{1/t}\theta^{-1}$. Let $B = B((x,y),r)$ with $x \in E', y \in F$ and $1 > r > M_\eta \delta_k = M_\eta r_1^k \theta$. Since $d(E) < 1$, it is enough to consider $r < 1$. Then for $E'_{k,i} = S_{i_1} \circ ... \circ S_{i_l}(E')$,

$$E_{k,i} \cap B \subset E'_{k,i} \times (F \cap B(y,r)).$$

Thus by (22), the definitions of μ, M_η and c_3, and by (24),

$$\mu(E_{k,i} \cap B) \le H^t(E'_{k,i}) H^{s-t}(F \cap B(y,r)) \le (r_{i_1}...r_{i_l})^t H^t(E') c_1 r^{s-t}$$

$$\le r_1^{kt} H^t(E') c_1 r^{s-t} \le (M_\eta\theta)^{-t} r^t H^t(E') c_1 r^{s-t}$$

$$= \eta c_3^{-1} r^s \le \eta\mu(B).$$

Hence (6) holds and Theorem 2.1 gives $\mu(E \cap f(A)) = 0$. By (24) μ and $H^s \mid E$ are comparable, as well as $H^s(E \cap f(A)) = 0$.

3. Self-similar sets with disjoint parts

In this section we give different proofs for two special cases of Theorem 2.1. First, the proof of the following theorem will be quite simple and elementary.

Theorem 3.1. *Let $E \subset \mathbb{R}^n$ be a compact self-similar set, $E = \bigcup_{i=1}^N S_i(E)$, $N \geq 2$, such that $S_i(E) \cap S_j(E) = \emptyset$ for $i \neq j$. If $s = \dim E$ and $f : \mathbb{R} \to \mathbb{R}^n$ is a $(1/s)$-Hölder map, then*

$$H^s(E \cap f(\mathbb{R})) = 0.$$

Proof. We may assume $N \geq 3$; otherwise we replace the similarities S_i with all $S_i \circ S_j$. To simplify the argument we assume that all the similarities $S_1, ..., S_N$ have the same similarity ratio r. In the case where they are different we could cut the sequences $i_1, i_2, ...$ so that $r_{i_1} ... r_{i_l} \sim r_1^k$, as in section 2, and this would complicate the argument only slightly. If the above statement was not true, we could find $f : I \to \mathbb{R}^n, I = [a, b]$, such that $\mid f(x) - f(y) \mid \leq \mid x - y \mid^{1/s}$ for $x, y \in I$ and $H^s(E \cap f(I)) > 0$.

Let $E_{k,i}, i = 1, ..., N^k$, be all the sets $S_{i_1} \circ ... \circ S_{i_k}(E)$. It can easily be shown that there is $E_0 = S_{i_1} \circ ... \circ S_{i_{k_0}}(E)$ such that for all $k > k_0$,

$$\text{card}\{i : E_{k,i} \cap f(I) \neq \emptyset, E_{k,i} \subset E_0\} > \frac{1}{2} N^{k-k_0}. \tag{26}$$

Let $F_{k,i}, k = 1, ..., m_k$, $k > k_0$, be all the sets $E_{k,j} \subset E_0$ such that $E_{k,j} \cap f(I) \neq \emptyset$. Then by (26), $m_k > (1/2)N^{k-k_0} \geq N^{k-k_0-1} = N_k$. For $k > k_0$ we can find N_k open, disjoint subintervals $I_j^k = (a_j^k, b_j^k), j = 1, ..., N_k$, such that $f(a_j^k) \in E_{k,j_1}$, $f(b_j^k) \in E_{k,j_2}$, $j_1 \neq j_2$, and $f(I_j^k) \cap E_0 = \emptyset$. Letting

$$\theta = \min\{d(S_i(E), S_j(E)) : i \neq j\},$$

we have for $j_1 \neq j_2$,

$$\theta r^k \leq d(E_{k,j_1}, E_{k,j_2}) \leq \mid f(a_j^k) - f(b_j^k) \mid \leq \mid a_j^k - b_j^k \mid^{1/s}. \tag{27}$$

All the different intervals $I_i^k, i = 1, ..., N_k, k = k_0 + 1, k_0 + 2, ...,$ are disjoint by their construction. But we may have $I_i^k = I_j^l$ for some $k \neq l$. However, for $k > k_0 + 1$,

$$\begin{aligned} &\text{card}\{I_i^k : I_i^k \notin \{I_j^l : j = 1, ..., N_l, l = k_0 + 1, ..., k - 1\}\} \\ &\geq N_k - \sum_{l=k_0+1}^{k-1} N_l = N^{k-k_0-1} - \sum_{l=k_0+1}^{k-1} N^{l-k_0-1} \\ &= N^{k-k_0-1} - (N^{k-k_0-1} - 1)/(N - 1) \geq \frac{1}{2} N^{k-k_0-1}, \end{aligned}$$

since $N \geq 3$. Letting $J_i^k, i = 1, \ldots, M_k$, where $M_k \geq (1/2)N^{k-k_0-1}$, be all the intervals I_i^k which do not agree with any I_j^l for $l < k$, we get from (27) for any $k > k_0 + 1$, recalling that $Nr^s = 1$,

$$b - a \geq \sum_{l=k_0+1}^{k} \sum_{j=1}^{M_l} \mathcal{L}^1(J_j^l) \geq \sum_{l=k_0+1}^{k} \frac{1}{2} N^{l-k_0-1} \theta^s r^{ls} = \frac{1}{2} N^{-k_0-1} \theta^s (k - k_0 - 1).$$

Letting $k \to \infty$ we get a contradiction which completes the proof.

The proof of our second special case of Theorem 2.1 will depend on the following lemma. This lemma reduces the proof to m-dimensional sets and Lipschitz maps from $\mathbb{R}^m$. We leave its easy natural proof to the reader.

Lemma 3.2. *Let E and E' be compact self-similar subsets of $\mathbb{R}^n$ of dimensions s and s', respectively. Furthermore, we assume that E and E' are of the form*

$$E = \bigcup_{i=1}^{N} S_i(E), \;\; E' = \bigcup_{i=1}^{N} S_i'(E'),$$

where the unions are disjoint, $r_1, \dots, r_N, r_1', \dots, r_N'$ are the similarity ratios of $S_1, \dots, S_N, S_1', \dots, S_N'$, and $r_i^s = (r_i')^{s'}$ for $i = 1, \dots, N$. Then there is a bijective (s/s')-Hölder map $g : E \to E'$ such that g^{-1} is (s'/s)-Hölder.

Remark. Lemma 3.2 is not valid for arbitrary totally disconnected self-similar sets, even in $\mathbb{R}$; see [FM].

Suppose now that $E = \bigcup_{i=1}^{N} S_i(E)$ is a compact self-similar set such that the sets $S_i(E)$ are disjoint and the open set condition is satisfied with an open convex set O. That is, there is a non-empty open convex set O such that

$$\bigcup_{i=1}^{N} S_i(O) \subset O \text{ and } S_i(\bar{O}) \cap S_j(\bar{O}) = \emptyset \text{ for } i \neq j.$$

(We may assume $S_i(\bar{O}) \cap S_j(\bar{O}) = \emptyset$ instead of $S_i(O) \cap S_j(O) = \emptyset$ since the sets $S_i(E)$ are disjoint.)

Theorem 3.3. *If E is as above, $s = \dim E > m$, $A \subset \mathbb{R}^m$, and $f : A \to \mathbb{R}^n$ is an (m/s)-Hölder map, then*

$$H^s(E \cap f(A)) = 0.$$

Proof. Let S_i be given by

$$S_i(x) = g_i(r_i(x - a_i)) + a_i,$$

where $g_i \in O(n)$ is a rotation and a_i is the fixed point of S_i. Then $\sum r_i^s = 1$, whence $\sum_{i=1}^{m} r_i^m > 1$ and we can find $r < 1$ such that $\sum_{i=1}^{m} (rr_i)^m = 1$.

Define S_i^* by

$$S_i^*(x) = g_i(rr_i(x - a_i)) + a_i$$

and let E^* be the self-similar set defined by $E^* = \bigcup_{i=1}^{N} S_i^*(E^*)$. Since $a_i \in E \subset \bar{O}$, one easily checks that $S_i^*(O) \subset S_i(O)$. Thus E^* satisfies the open set condition, $0 < H^m(E^*) < \infty$ and E^* has disjoint parts, as $S_i^*(E^*) \subset S_i^*(\bar{O}) \subset S_i(\bar{O})$ and the sets $S_i(\bar{O})$ are disjoint.

By Lemma 3.2 there is a bijective (s/m)-Hölder map $g : E \to E^*$ whose inverse $g^{-1} : E^* \to E$ is (m/s)-Hölder. Let $f : A \to \mathbb{R}^n, A \subset \mathbb{R}^m$, be an (m/s)-Hölder map. Replacing f by $f \mid A \cap f^{-1}(E)$ we may assume $f : A \to E$. Then $g \circ f : A \to \mathbb{R}^n$ is a Lipschitz map. The m-dimensional self-similar sets with disjoint parts are purely m-unrectifiable (see [H]). Hence $H^m((g \circ f)(A) \cap E^*) = 0$. However,

$$g^{-1}((g \circ f)(A) \cap E^*) = f(A) \cap E.$$

Since g^{-1} is (m/s)-Hölder, it follows that $H^s(f(A) \cap E) = 0$.

If we allow the Hölder exponent to be less than $m/\dim E$, we can easily find such Hölder maps whose image covers E for a large class of sets E. The following result can be proved by the method of [MM1, Theorem 4.1(1)].

Theorem 3.4. *Suppose m and n are positive integers, $m \leq s < n, E \subset \mathbb{R}^n$ is H^s measurable and there is a positive number c such that*

$$r^s/c \leq H^s(E \cap B(x,r)) \leq cr^s \text{ for } x \in E \text{ and } 0 < r < 1.$$

If $0 < \alpha < m/s$, there is an α-Hölder map $f : \mathbb{R}^m \to \mathbb{R}^n$ such that $E \subset f(\mathbb{R}^m)$.

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