PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 9, Pages 2659–2665
S 0002-9939(00)05389-2
Article electronically published on February 28, 2000

NON–CONTINUATION OF THE PERIODIC OSCILLATIONS OF A FORCED PENDULUM IN THE PRESENCE OF FRICTION

RAFAEL ORTEGA, ENRICO SERRA, AND MASSIMO TARALLO

(Communicated by Hal L. Smith)

Abstract. A well known theorem says that the forced pendulum equation has periodic solutions if there is no friction and the external force has mean value zero. In this paper we show that this result cannot be extended to the case of linear friction.

1. Introduction

The periodic problem for the forced pendulum equation has been studied by many authors since the initial work [5]. In that paper Hamel considered the problem

$$\begin{cases} \ddot{x}(t) + a\sin x(t) = \beta \sin t, \\ x(t+2\pi) = x(t) \qquad\qquad \forall t \end{cases}$$

and proved, among other results, that it has a solution which can be obtained by minimization of the action functional

$$J[x] = \int_0^{2\pi} \{\frac{1}{2}\dot{x}(t)^2 + a\cos x(t) + x(t)\beta\sin t\}dt, \quad x(t+2\pi) = x(t).$$

The same argument can be extended to the more general equation

$$\ddot{x} + a\sin x = h(t), \tag{1}$$

where h is a T-periodic function satisfying

$$h \in L^1(\mathbf{R}/T\mathbf{Z}), \quad \int_0^T h(t)dt = 0. \tag{2}$$

It leads to the following result.

Theorem 1.1. *Equation* (1) *has at least one* T*-periodic solution if* (2) *holds.*

This result is a simple and elegant application of the variational method (see [3, 6] for a proof and [8] for more information on this problem).

Let us now consider the equation with friction

$$\ddot{x} + c\dot{x} + a\sin x = h(t), \tag{3}$$

where $c > 0$. In this case variational methods do not seem applicable to the periodic problem and Mawhin asked in [6] and [7] whether or not it was possible to extend

Received by the editors October 22, 1998.
1991 *Mathematics Subject Classification.* Primary 34C25.

Theorem 1.1 to (3). In [10] it was found that the answer to this question is negative. In fact there exist functions h satisfying (2) and such that (3) has no periodic solutions. The construction in [10] required extra assumptions on the parameters a, T and c. In particular, it was assumed that a and c were very large. In a more recent paper Alonso [1] has constructed new counterexamples that partially remove these restrictions. In [1] the positive constants a and c are arbitrary but T has to be sufficiently large. After these works it was still possible to ask whether Theorem 1.1 could have an extension to (3) for some special values of the parameters. The purpose of this note is to construct a rather explicit class of counterexamples that are valid for arbitrary values of the parameters a, c and T. To state the result in a precise way we need to introduce some notation.

Given a function h satisfying (2), H is the unique solution of

$$\ddot{H} + c\dot{H} = h(t), \quad H(t+T) = H(t), \quad \int_0^T H(t)dt = 0.$$

The norm $||.||$ will always be the L^2 norm on periodic functions; that is,

$$||f|| = (\int_0^T f(t)^2 dt)^{1/2}, \quad f \in L^2(\mathbf{R}/T\mathbf{Z}).$$

Given $T > 0$ the function $\mathcal{B}_T$ is defined as

$$\mathcal{B}_T(t) = 2\pi(\frac{t}{T} - [\frac{t}{T}]),$$

where $[s]$ denotes the largest integer smaller than or equal to $s \in \mathbf{R}$. This function belongs to $L^2(\mathbf{R}/T\mathbf{Z})$.

Theorem 1.2. *Given positive constants a, c and T, there exists $\epsilon > 0$ such that the equation* (3) *has no T-periodic solutions if h satisfies* (2) *and*

$$||H - \mathcal{B}_T|| < \epsilon.$$

Moreover, ϵ can be explicitly computed in terms of a, c and T.

This result, as well as the results in [10, 1], shows how natural is the variational method in the proof of Theorem 1.1. We now present a dynamical consequence.

Corollary 1.3. *In the conditions of Theorem 1.2 every solution of* (3) *satisfies*

$$\lim_{t \to +\infty} |x(t)| = \infty.$$

To conclude this introduction we derive a curious mechanical consequence which follows from Theorem 1.1 and Corollary 1.3. Let us think of equations (1) and (3) as models for the motion of a pendulum under the action of a torque with variable moment $h(t)$. The condition (2) says that the averaged moment is zero. Let us now imagine two particles, $\mathcal{C}$ and $\mathcal{F}$, which move on the circle according to (1) and (3) respectively. Except for the friction factor, they are under the same physical laws, and it will also be assumed that they satisfy the same initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$. Then we can find an external moment h and appropriate initial conditions in such a way that the particle under friction $\mathcal{F}$ will experience an infinite number of revolutions ($\lim_{t\to+\infty} |x(t)| = \infty$) while the conservative particle $\mathcal{C}$ will oscillate periodically and, in particular, $\limsup_{t\to+\infty} |x(t)| < \infty$.

2. A preliminary result

We consider the problem

$$\begin{cases} \ddot{x}(t) + c\dot{x}(t) + a\sin x(t) = h(t), \\ x(t+T) = x(t) & \forall t, \end{cases} \tag{4}$$

where $c \geq 0$ and h satisfies (2). We denote from now on

$$\omega = \frac{2\pi}{T}.$$

We start by making the classical change of variable given by

$$y(t) = x(t) - H(t),$$

where H is the function associated to h as in the introduction. Replacing x with $y + H$ in (4) transforms the problem into

$$\begin{cases} \ddot{y}(t) + c\dot{y}(t) + a\sin(y(t) + H(t)) = 0, \\ y(t+T) = y(t) & \forall t. \end{cases} \tag{5}$$

The set of solutions of problem (5) is in one–to–one correspondence with the set of solutions of (4). Since h satisfies (2) the function H belongs to $W^{2,1}(\mathbf{R}/T\mathbf{Z})$. If H is less regular, problem (5) still makes sense, but is no longer equivalent to problem (4). In fact (5) is well defined for any function H in $L^1(\mathbf{R}/T\mathbf{Z})$ and, in particular, when $H = \mathcal{B}_T$. Due to the periodicity of the sine function, problem (5) for $h = \mathcal{B}_T$ is equivalent to

$$\begin{cases} \ddot{y}(t) + c\dot{y}(t) + a\sin(y(t) + \omega t) = 0, \\ y(t+T) = y(t) & \forall t. \end{cases} \tag{6}$$

This problem was already considered by Bates in [2]. He proved that if $c = 0$, there exists a continuum of solutions (see also [11, 12]). The next result shows that the situation is different when friction is present.

Proposition 2.1. *Assume that the friction coefficient is positive; that is,*

$$c > 0.$$

Then (6) *has no T-periodic solutions.*

Proof. By a contradiction argument assume that $y(t)$ is a solution of (6). Multiplying the equation by $\dot{y} + \omega$ and integrating over one period, there results

$$c\int_0^T \dot{y}^2\, dt = 0 \Rightarrow y \equiv constant,$$

which is impossible. □

The proof of Theorem 1.2 will be based on a "perturbation" of the previous argument that is valid for (5).

3. Proofs

In what follows we shall consider problem (5) and we will decompose H in the form $H(t) = \mathcal{B}_T(t) + P(t)$. We will show that (5) has no solutions whenever $||P||$ is small enough. We will give an explicit estimate on $||P||$ so that (5) has this property.

The proof is based on two lemmas which follow along the lines of the previous section. In the first lemma we will show that if y solves (5) with $H = \mathcal{B}_T + P$ as above, then $||\dot{y}||$ must be small if $||P||$ is small. Then we show that the same problem cannot have solutions y with small $||\dot{y}||$. As before, we can, and will, write $\sin(y(t) + \mathcal{B}_T(t) + P(t))$ as $\sin(y(t) + \omega t + P(t))$.

Lemma 3.1. *Let y be a T–periodic solution of*

$$\ddot{y} + c\dot{y} + a\sin(y + \omega t + P(t)) = 0 \tag{7}$$

with $P \in L^2(\mathbf{R}/T\mathbf{Z})$. Then

$$||\dot{y}||^2 \le \varphi(\frac{a}{c}||P||),$$

where

$$\varphi(s) = \frac{1}{2}\left(s^2 + \sqrt{s^4 + \frac{16\pi^2}{T}s^2}\right).$$

Proof. We add and subtract the term $a\sin(y + \omega t)$ in equation (7), we multiply it by $\dot{y} + \omega$ and we integrate over $[0, T]$: we obtain

$$\begin{aligned} c||\dot{y}||^2 &= a\int_0^T [\sin(y + \omega t) - \sin(y + \omega t + P)]\,(\dot{y} + \omega)\,dt \\ &\le a\int_0^T |P(t)||\dot{y} + \omega|\,dt \ \le\ a\,||P||\,||\dot{y} + \omega||. \end{aligned}$$

Squaring gives

$$c^2||\dot{y}||^4 \le a^2||P||^2\left(||\dot{y}||^2 + \frac{4\pi^2}{T}\right),$$

or

$$||\dot{y}||^4 - \frac{a^2}{c^2}||P||^2\,||\dot{y}||^2 - \frac{a^2}{c^2}\frac{4\pi^2}{T}||P||^2 \le 0. \tag{8}$$

Setting $s = \frac{a}{c}||P||$ we see that in order for (8) to hold it must be

$$||\dot{y}||^2 \le \frac{1}{2}\left(s^2 + \sqrt{s^4 + \frac{16\pi^2}{T}s^2}\right),$$

and the statement is proved. □

We now prove a reversed estimate.

Lemma 3.2. *Let y be a T–periodic solution of (7) with $P \in L^2(\mathbf{R}/T\mathbf{Z})$. Then*

$$||\dot{y}||^2 \ge \frac{Ta^2}{2(\omega^2 + c^2)} - 2\frac{\sqrt{T}a^2}{\omega^2 + c^2}\left(\frac{1}{\omega^2}\varphi(\frac{a}{c}||P||) + ||P||^2\right)^{\frac{1}{2}}.$$

Proof. The average of a T-periodic function $x = x(t)$ will be denoted by

$$\overline{x} := \frac{1}{T}\int_0^T x(t)dt.$$

Thus $\overline{y}$ is the average of the solution y and we can rewrite equation (7) as

$$\ddot{y} + c\dot{y} + a\sin(\overline{y} + \omega t) = a\sin(\overline{y} + \omega t) - a\sin(y + \omega t + P(t)). \tag{9}$$

We call $f(t)$ the right–hand side of (9). Note that $\overline{f} = 0$ if y solves (9) so that we can define F to be the unique solution of the problem

$$\begin{cases} \ddot{F} + c\dot{F} = f, \\ F(t+T) = F(t) \quad \forall t, \\ \overline{F} = 0. \end{cases}$$

We now change the variable in (9) by setting $z(t) = y(t) - F(t)$: we obtain that (9) reduces to

$$\ddot{z} + c\dot{z} + a\sin(\overline{y} + \omega t) = 0.$$

Now, with straightforward computations we see that there results

$$z(t) = \frac{a}{\omega^2 + c^2}\sin(\overline{y} + \omega t) + \frac{ac}{\omega(\omega^2 + c^2)}\cos(\overline{y} + \omega t)$$

and

$$||\dot{z}||^2 = \frac{Ta^2}{2(\omega^2 + c^2)}$$

so that

$$\begin{aligned} ||\dot{y}||^2 &= ||\dot{z}||^2 + ||\dot{F}||^2 + 2\int_0^T \dot{z}\dot{F}\,dt \;\geq\; ||\dot{z}||^2 - 2||\dot{z}||\,||\dot{F}|| \\ &= \frac{Ta^2}{2(\omega^2 + c^2)} - \sqrt{\frac{2Ta^2}{\omega^2 + c^2}}||\dot{F}||. \end{aligned} \tag{10}$$

We now evaluate $||\dot{F}||$ by expanding f and F in Fourier series: we easily obtain

$$||\dot{F}||^2 \leq \frac{||f||^2}{\omega^2 + c^2}. \tag{11}$$

Finally we evaluate $||f||$. By its very definition we have

$$|f(t)| \leq a\,|y(t) - \overline{y} + P(t)| \leq a(|y(t) - \overline{y}| + |P(t)|).$$

Squaring and integrating yields, by the Wirtinger inequality,

$$||f||^2 \leq 2a^2\int_0^T \left(|y - \overline{y}|^2 + |P|^2\right)\,dt \leq 2a^2\left(\frac{1}{\omega^2}||\dot{y}||^2 + ||P||^2\right).$$

Inserting this estimate in (11) and then in (10) we conclude that

$$||\dot{y}||^2 \geq \frac{Ta^2}{2(\omega^2 + c^2)} - \frac{2\sqrt{T}a^2}{\omega^2 + c^2}\left(\frac{1}{\omega^2}||\dot{y}||^2 + ||P||^2\right)^{\frac{1}{2}}.$$

Finally we apply Lemma 3.1 to obtain the conclusion. ◻

Proof of Theorem 1.2. If

$$\varphi(\frac{a}{c}||P||) + \frac{2\sqrt{T}a^2}{\omega^2 + c^2}\left(||P||^2 + \frac{1}{\omega^2}\varphi(\frac{a}{c}||P||)\right)^{\frac{1}{2}} < \frac{Ta^2}{2(\omega^2 + c^2)}, \tag{12}$$

then problem (5) has no solution. Indeed, since $\varphi(s) \to 0$ as $s \to 0$, the two estimates provided by Lemma 3.1 and Lemma 3.2 are incompatible as soon as $||P||$ is so small that inequality (12) holds. Note also that by means of (12), for every a and c we can estimate *explicitly* the threshold for $||P||$ under which problem (5) has no solution. □

Proof of Corollary 1.3. It is sufficient to prove that all solutions of the equation

$$\ddot{y} + c\dot{y} + a\sin(y + H(t)) = 0$$

satisfy $|y(t)| \to \infty$ as $t \to +\infty$.

Let $\mathcal{P}$ be the Poincaré mapping associated to this differential equation. It is defined by

$$(y(0), \dot{y}(0)) \mapsto (y(T), \dot{y}(T)),$$

where $y(t)$ is an arbitrary solution. It is well known that $\mathcal{P}$ is an orientation preserving homeomorphism of the plane and Theorem 1.2 implies that it does not have fixed points. Thus, the theory of Brouwer of fixed point free homeomorphisms is applicable. (See for instance [4, 9]). This theory implies that the orbits of $\mathcal{P}$ must go to infinity; that is,

$$|(y(nT), \dot{y}(nT))| \to \infty, \quad \text{as } n \to +\infty,$$

for every solution $y(t)$.

On the other hand we can prove that the derivative $\dot{y}(t)$ is bounded. In fact, $v = \dot{y}$ satisfies the differential inequality

$$|\dot{v} + cv| \leq a,$$

implying

$$|v(t)| \leq e^{-ct}|v(0)| + \frac{a}{c}(1 - e^{-ct}), \quad \forall t \geq 0.$$

Define $M = \sup_{t\geq 0}|\dot{y}(t)|$. The sequence $|\dot{y}(nT)|$ is bounded by M and so $|y(nT)| \to \infty$. Finally, from the mean value theorem, we deduce that for each $t \in [nT, (n+1)T]$,

$$|y(t)| \geq |y(nT)| - MT.$$

This implies that $|y(t)|$ goes to infinity. □

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Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain
E-mail address: rortega@goliat.ugr.es

Dipartimento di Matematica del Politecnico, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
E-mail address: serra@polito.it

Dipartimento di Matematica dell'Università, Via Saldini 50, 20133 Milano, Italy
E-mail address: tarallo@vmimat.mat.unimi.it