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ON SOME PROPERTIES OF THE GAMMA FUNCTION

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ABSTRACT. Anderson and Qiu (1997) conjectured that the function $\frac{\log\Gamma(x+1)}{x\log x}$ is concave for $x>1$. In this paper we prove this conjecture. We also study the monotonicity of some functions connected with the psi-function $\psi(x)$ and derive inequalities for $\psi(x)$ and $\psi'(x)$.

1. Introduction

For $x>0$ let $\Gamma(x)$ and $\psi(x)$ denote the Euler's gamma function, defined by

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt, \qquad \psi(x)=\frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. There is a vast literature on these functions and a good reference to this can be found, for example, in the recent paper [2].

Anderson and Qiu showed that the function $\frac{\log\Gamma(x+1)}{x\log x}$ strictly increases from $1-\gamma$ to 1 as x increases from 1 to ∞, where $\gamma=0.577...$ denotes the Euler-Mascheroni constant. To do this, they investigated the function

$$f(x)=\psi'(1+x)+x\psi''(1+x) \tag{1.1}$$

and they found the representation

$$f(x)=\sum_{n=1}^\infty \frac{n-x}{(n+x)^3}. \tag{1.2}$$

They proved, in a complicated way, that $f(x)>0$ for $x\in[1,4)$ and formulated the following:

Conjecture. *The function* $\frac{\log\Gamma(x+1)}{x\log x}$ *is concave for* $x>1$.

In Section 2 we extend the inequality $f(x)>0$ from $[1,4)$ to $(-1,\infty)$ (this extension is evident for $-1<x\le 1$). Then we derive also new inequalities for $\psi(x)$ and $\psi'(x)$.

In Section 3 we prove the conjecture formulated above by Anderson and Qiu [3].

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We recall now the following two asymptotic representations [1, p. 260; 6.4.12, 6.4.13]:

$$\psi'(z) \sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} + \cdots \qquad (z \to \infty, |\arg z| < \pi), \tag{1.3}$$

$$\psi''(z) \sim -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2z^4} + \frac{1}{6z^6} - \cdots \qquad (z \to \infty, |\arg\ z| < \pi), \tag{1.4}$$

which will be used only for real z's in the next sections.

2. New results

Our first result provides an extension of one proved by Anderson and Qiu [3]. Indeed, we prove, in a simple way, that $f(x)$ is positive not only for $x \geq 1$, but even for $x > -1$.

Theorem 1. *Let $f(x)$ be defined by* (1.1)*. Then $f(x) > 0$ for any $x > -1$.*

Proof. By the asymptotic formulas (1.3) and (1.4), we get

$$\lim_{x \to +\infty} f(x) = 0.$$

So, in order to show that $f(x) > 0$, it is sufficient to show that $f(x) - f(x+1) > 0$. By (1.2) we get

$$f(x) - f(x+1) = -\frac{1}{(1+x)^2} + \sum_{n=1}^{\infty} \frac{2}{(n+x)^3}.$$

Hence, by the inequality

$$\frac{2}{u^3} > \frac{1}{u^2} - \frac{1}{(1+u)^2}, \qquad u > 0,$$

we find that

$$f(x) - f(x+1) > -\frac{1}{(1+x)^2} + \sum_{n=1}^{\infty} \left[\frac{1}{(n+x)^2} - \frac{1}{(n+x+1)^2} \right] = 0.$$

The proof is complete. □

Now we use this result to derive the following

Theorem 2. *Let the function $g(x)$ be defined by*

$$g(x) = x^2 \psi'(1+x) - x\psi(1+x) + \log \Gamma(x+1), \qquad x > -1.$$

Then $g(x)$ strictly decreases from ∞ to 0 on $(-1, 0]$ and strictly increases from 0 to ∞ on $[0, \infty)$.

Proof. By differentiation we get

$$g'(x) = x\psi'(1+x) + x^2 \psi''(1+x) = xf(x),$$

where $f(x)$ is defined in (1.1).

By Theorem 1, $f(x) > 0$ for $x > -1$; hence $g(x)$ is decreasing for $-1 < x \le 0$ and is increasing for $x \ge 0$. Clearly $g(0) = 0$, and by [1, 6.3.16] we get

$$\begin{aligned} g(x) &= x^2\psi'(1+x) - x\psi(1+x) + \log\Gamma(1+x) \\ &= x^2\sum_{n=1}^{\infty}\frac{1}{(n+x)^2} - x\left[-\gamma + \sum_{n=1}^{\infty}\frac{x}{n(n+x)}\right] + \log\Gamma(1+x) \\ &= \sum_{n=1}^{\infty}\frac{-x^3}{n(n+x)^2} - \gamma x + \log\Gamma(1+x), \end{aligned}$$

which tends to ∞ as x tends to -1^+.

By [1, 6.3.5] and [1, 6.1.15] we get

$$g(x) - \frac{1}{2}\log x = \left[x^2\psi'(x) - x\right] + x\left[\log x - \psi(x)\right] + \log\left(\Gamma(x)e^x x^{-x+1/2}\right).$$

By [1, 6.4.12] the function in the first bracket tends to 1/2, and by Stirling's Formula [1, 6.1.37] the function in the third bracket tends to $\sqrt{2\pi}$, as x tends to ∞. Finally by [2, (2.1)], the middle term tends to 1/2 as x tends to ∞. So $g(x) - \frac{1}{2}\log x = \mathrm{O}(1)$ as x tends to ∞.

This completes the proof of Theorem 2. □

Theorem 3. *Let*

$$h(x) = x^2\psi'(x+1) + x^3\psi''(x+1)$$

for $x > 0$. *Then*

$$0 < h(x) < \frac{1}{2}.$$

Proof. The lower bound follows by Theorem 1 since $h(x) = x^2 f(x)$. To prove the upper bound we define the function $r(x)$ by

$$r(x) = \frac{1}{2x^2} - f(x), \qquad x > 0.$$

Clearly $r(x) \to 0$ as $x \to \infty$. Thus it is sufficient to show that $r(x) > r(x+1)$. Making use of the inequality

$$\frac{2}{u^3} < \frac{1}{2(u-1)^2} - \frac{1}{2(u+1)^2}, \qquad u > 1,$$

we obtain

$$\begin{aligned} r(x) - r(x+1) &= \frac{1}{2x^2} + \frac{1}{2(1+x)^2} - \sum_{n=1}^{\infty}\frac{2}{(n+x)^3} \\ &> \frac{1}{2x^2} + \frac{1}{2(1+x)^2} - \sum_{n=1}^{\infty}\left[\frac{1}{2(n-1+x)^2} - \frac{1}{2(n+1+x)^2}\right] \\ &= \frac{1}{2x^2} + \frac{1}{2(1+x)^2} - \left[\frac{1}{2x^2} + \frac{1}{2(1+x)^2}\right] = 0. \end{aligned}$$

The proof of Theorem 3 is complete. □

Remark. The inequality $h(x) < \frac{1}{2}$, proved in Theorem 3, is equivalent to

$$\left(x\psi'(1+x) + \frac{1}{2x}\right)' < 0$$

or to

$$\left(x\psi'(x) - \frac{1}{2x}\right)' < 0.$$

Hence

$$x\psi'(x) - \frac{1}{2x} > \lim_{x\to\infty}\left(x\psi'(x) - \frac{1}{2x}\right) = 1$$

which leads to the inequality

$$\psi'(x) - \frac{1}{x} - \frac{1}{2x^2} > 0.$$

This inequality is the same that we obtain using the fact that the second derivative of the function

$$S_0(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x + x - \frac{1}{2}\log(2\pi)$$

is positive, a result which follows from the complete monotonicity property of $S_0(x)$ proved by Muldoon [5, Theorem 8].

3. Proof of the conjecture

We are going to prove that

$$\left(\frac{\log\Gamma(x+1)}{x\log x}\right)'' < 0, \qquad \text{for } x > 1. \tag{3.1}$$

To this end we distinguish two cases:

a) $1 < x \le 2$;

b) $x > 2$.

Proof of (3.1) in case a). We consider the infinite product representation [4, p. 37]

$$\Gamma(1+x) = \prod_{n=1}^{\infty}\frac{(1+\frac{1}{n})^x}{1+\frac{x}{n}}, \qquad x \ne -1, -2, \cdots.$$

Taking the logarithm, we find

$$\frac{\log\Gamma(x+1)}{x\log x} = \sum_{n=1}^{\infty} f\left(x, \frac{1}{n}\right)$$

where

$$f(x,t) = \frac{x\log(1+t) - \log(1+tx)}{x\log x}, \qquad 0 \le t \le 1.$$

If for some x we prove that

$$\frac{\partial^2}{\partial x^2} f\left(x, \frac{1}{n}\right) < 0, \qquad \text{for } n = 1, 2, \cdots,$$

then the inequality (3.1) holds for that value of x. Since $f(x,0) \equiv 0$, we have $\frac{\partial^2}{\partial x^2} f(x,0) \equiv 0$. We are going to show the inequality

$$\frac{\partial^2}{\partial x^2} f(x,t) < 0, \qquad \text{for } 1 \le x \le 2,\ 0 < t \le 1.$$

To this end we show that $\frac{\partial^2}{\partial x^2} f(x,t)$ is a decreasing function of t, at least for $1 \le x \le 2$, x fixed.

Indeed, we have

$$\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}f(x,t) = \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}f(x,t) = \frac{\partial^2}{\partial x^2}\frac{tx-t}{(1+t)(1+tx)\log x}$$
$$= \frac{t}{t+1}\frac{D(x,t)}{x^2(1+xt)^3\log^3 x},$$

where

$$D(x,t) = a(x) + b(x)t + c(x)t^2,$$
$$a(x) = -2 + 2x - (x+1)\log x,$$
$$b(x) = 2x(2x - 2 - 2\log x - x\log^2 x),$$
$$c(x) = x^2(2x - 2 - 3\log x + x\log x - 2\log^2 x).$$

We observe that $a(x) < 0$ and $c(x) > 0$ for $x > 1$. Indeed $a(1) = 0$ and $a'(x) < 0$ and therefore $a(x) < 0$, for $x > 1$. Concerning $c(x)$ we have $c(1) = 0$ and $\left(\frac{c(x)}{x^2}\right)' > 0$. We have to show that $D(x,t) < 0$. The function $D(x,t)$ is convex with respect to t; therefore we have only to show that $D(x,0) < 0$ and $D(x,1) < 0$.

For $t = 0$ we get $D(x,0) = a(x) < 0$. Moreover

$$\begin{aligned} d(x) &= D(x,1) = a(x) + b(x) + c(x) \\ &= 2(x+1)^2(x-1) + (x^3 - 3x^2 - 5x - 1)\log x - 4x^2\log^2 x \end{aligned} \tag{3.2}$$

and we need to show that $d(x) < 0$. It is easy to check that $d(1) = 0$, $d(2) = 18 - 15\log 2 - 16\log^2 2 = -0.084... < 0$. By (3.2)

$$\left(\frac{d(x)}{x^2}\right)' = \frac{x-1}{x^3}[3x^2 - 3 + (x^2 - 7x - 2)\log x] = \frac{x-1}{x^3}d_1(x),$$

where $d_1(1) = 0$, $d_1(2) = 9 - 12\log 2 = 0.682... > 0$, and

$$\left(\frac{d_1(x)}{2+7x-x^2}\right)' = -\frac{d_2(x)}{x(2+7x-x^2)^2},$$

where

$$d_2(x) = x^4 - 35x^3 + 39x^2 + 7x + 4.$$

Since

$$d_2(1) = 16 > 0,$$
$$d_2(2) = -90 < 0,$$
$$\lim_{x\to\infty} d_2(x) = +\infty,$$

it follows that $d_2(x)$ has a zero in $(1,2)$ and a zero in $(2,\infty)$. By Descartes' Rule of Signs $d_2(x)$ can have either no positive zeros or two positive zeros. Thus we conclude that d_2 has exactly two positive zeros. Consequently $d_2(x)$ has only one zero, say ξ_2, in $(1,2)$. Thus the function $\left(\frac{d_1(x)}{2+7x-x^2}\right)'$ is negative in $(1,\xi_2)$ and positive in $(\xi_2,2)$, and in particular $d_1(\xi_2) < 0$. Since also $d_1(2) > 0$, we see that $d_1(x)$ has exactly one zero ξ_1 in $(1,2)$. Finally we have that $\left(\frac{d(x)}{x^2}\right)'$ is negative in $(1, \xi_1)$ and positive in $(\xi_1, 2)$. Since also $d(1) = 0$ and $d(2) < 0$, this shows that $d(x) < 0$ in $(1, 2)$. This completes the proof of the conjecture for $x \in (1, 2]$. □

Proof of (3.1) in case b). Direct calculations show that

$$\left(\frac{\log\Gamma(x+1)}{x\log x}\right)'' = \frac{2}{x^3} + \frac{2+3l+2l^2}{x^3l^3}\log\Gamma(x) - 2\frac{l+1}{x^2l^2}\psi(x) + \frac{1}{xl}\psi'(x) \tag{3.3}$$

where $l = \log x$, and we have to prove that the right-hand side is negative for $x > 2$.

We are looking for an upper bound for $\log\Gamma(x)$, a lower bound for $\psi(x)$ and an upper bound for $\psi'(x)$. By [2, p. 383; Theorem 8] we get that the function

$$G_0(x) = -\log\Gamma(x) + \left(x - \frac{1}{2}\right)l - x + \lambda + \frac{1}{12x},$$

where

$$\lambda = \frac{\log(2\pi)}{2},$$

is strictly completely monotonic on $(0, \infty)$.

This gives that

$$\begin{aligned}
\log\Gamma(x) &< \left(x - \frac{1}{2}\right)l - x + \lambda + \frac{1}{12x},\\
\psi(x) &> l - \frac{1}{2x} - \frac{1}{12x^2},\\
\psi'(x) &< \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}.
\end{aligned}$$

Thus replacing $\log\Gamma(x)$, $\psi(x)$, $\psi'(x)$ in (3.3) with these bounds, we have

$$\begin{aligned}
x^3l^3\left(\frac{\log\Gamma(x+1)}{x\log x}\right)'' &< 2l^3 + (2+3l+2l^2)\left[\left(x - \frac{1}{2}\right)l - x + \lambda + \frac{1}{12x}\right]\\
&\quad - 2xl(l+1)\left[l - \frac{1}{2x} - \frac{1}{12x^2}\right] + x^2l^2\left(\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}\right)\\
&= A(l)x + B(l) + \frac{1}{x}C(l),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
A(l) &= -(l+2),\\
B(l) &= l^3 + 2\lambda l^2 + 3\lambda l + 2\lambda,\\
C(l) &= \frac{1}{2}l^2 + \frac{5}{12}l + \frac{1}{6}.
\end{aligned}$$

Let us consider the quadratic polynomial

$$Q(z) = A(l)z^2 + B(l)z + C(l).$$

By (3.4) we have to show that $Q(x) < 0$ for $x > 2$. Since $A(l) < 0$ and $C(l) > 0$, the polynomial $Q(z)$ has exactly one positive zero z_0 and $Q(z)$ is negative when $z > z_0$. The Taylor polynomial of the second degree for e^l at $l = \log 2$ gives a lower bound for e^l, $l > \log 2$. Thus we find

$$x = e^l > l^2 + \mu l + \nu \equiv z_1, \qquad x > 2, \tag{3.5}$$

where

$$\mu = 2 - 2\log 2, \qquad \nu = 1 + (1 - \log 2)^2.$$

Using the numerical values of μ and ν we obtain

$$Q(z_1) < -0.775828\, l^4 - 0.040884\, l^3 - 0.432207\, l^2 + 0.677828\, l - 0.216770$$

and we wish to show that $Q(z_1) < 0$ for $l > \log 2$. Using the inequalities

$$l^4 \geq l^2 \log^2 2, \qquad l^3 \geq l^2 \log 2,$$

we get

$$Q(z_1) < -0.833295\, l^2 + 0.677828\, l - 0.216770.$$

Since the polynomial on the right-hand side has no real zeros, we have $Q(z_1) < 0$. This shows that $z_1 > z_0$ and consequently $Q(x) < 0$, because by (3.5), $x > z_1$. This completes the proof of the conjecture. □

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