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ERRATA TO "ON SEMISIMPLE HOPF ALGEBRAS OF DIMENSION pq"

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We thank Sonia Natale who pointed out an error at the end of the proof of Theorem 3.4 in [GW]. We now wish to correct it. The corrected proof extends our original argument used in the case $\dim A = 3p$ (preprint, 1997).

First note that the "if and only if" statements preceding equations (3) and (4) are incorrect; the "if" direction is true but now unnecessary, so delete the lines after equation (2) to equation (5).

Now, the statement of Theorem 3.4 in [GW] should be replaced by the statement of Theorem 3.5 in [GW] (whose proof in turn should be deleted), and the corrected proof is as follows.

First, delete the first and third sentences of the third paragraph. Second, replace "Therefore by (5)," in the beginning of line 17 of the third paragraph by "Since the dimension of the intersection between a minimal left ideal and a minimal right ideal of a matrix algebra over k is 1, it follows that". Finally, delete the part starting with equation (10) and ending at the end of the proof, and replace it with the following:

Hence $\dim R_i = p$ and in particular $\alpha_{ij} \neq 0$ for all i, j. It is now straightforward to check that V^* is a commutative algebra, and hence that V is a cocommutative coalgebra. But since $A^* \cong B^* \times H$ (see [G]) is semisimple, B is cosemisimple and thus it follows that V has a basis consisting of grouplike elements. Therefore, since $B = k1 \oplus V_1 \oplus \cdots \oplus V_a$ where V_i is an irreducible left coideal of A of dimension p for all i, it follows that B is a cocommutative coalgebra. By symmetry, B^* is also cocommutative and hence B is a commutative algebra.

Let $\{x_{ij} | 1 \leq j \leq p\}$ be a basis of V_i consisting of grouplike elements. Then $\{1\} \cup \{x_{ij} | 1 \leq i \leq a,\ 1 \leq j \leq p\}$ is a basis of B consisting of grouplike elements.

We recall from Proposition 3.3 in [GW] and its proof that $\{C_i = V_i \times H | 1 \leq i \leq a\}$ is the set of all p^2-dimensional simple subcoalgebras of A, $\chi_i \in C_i$ is the corresponding irreducible character of A^*, $\{g^j, \chi_i | 0 \leq j \leq p-1,\ 1 \leq i \leq a\}$ is the set of all irreducible characters of A^* and $\chi_i g = g\chi_i = \chi_i$ for all i. Since $gC_i = C_i = C_i g$, V_i is an H-submodule of B.

In the following lemma we describe the action $\cdot$ of H on B.

Lemma 1. *For all i and j, we may assume that $g \cdot x_{ij} = x_{i,j+1}$.*

Proof. Since $\Delta_B(g \cdot b) = g \cdot \Delta_B(b)$ for all $b \in B$ [R], each $g \cdot x_{ij}$ is a grouplike element of V_i. Now for all i, g does not act trivially on V_i (otherwise g acts trivially on C_i

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and H is a normal sub-Hopf algebra of A (see e.g. the proof of [M, Claim 7])), and thus it follows that g necessarily permutes the basis $\{x_{ij}\}$ of V_i without any fixed points. $\square$

Fix a primitive pth root of unity $\omega \in k$, and let $\{E_m := \frac{1}{p}\sum_{t=0}^{p-1}\omega^{mt}g^t | 0 \le m \le p-1\}$ be the set of all the primitive idempotents of H. In the following lemma we describe the coaction $\rho : B \to H \otimes B$.

Lemma 2. *For all i and j, $\rho(x_{ij}) = \sum_{l=1}^{p} E_{j-l} \otimes x_{il}$.*

Proof. By Lemma 1, $\sum_{j=1}^{p}\omega^{-jt}x_{ij}$ spans the 1-dimensional eigenspace of the action by g on V_i corresponding to the eigenvalue ω^t. On the other hand, it follows by the compatibility condition that each homogeneous component $\rho^{-1}(g^t \otimes V_i)$ of V_i is stable under the action of g, and hence is spanned by an eigenvector of g. Now using the fact that V_i is an H-module coalgebra, it is straightforward to verify that $\rho^{-1}(g^t \otimes V_i)$ is the eigenspace corresponding to ω^t. We thus conclude that $\rho(\sum_{j=1}^{p}\omega^{-jt}x_{ij}) = g^t \otimes (\sum_{j=1}^{p}\omega^{-jt}x_{ij})$. Since $x_{ij} = \frac{1}{p}\sum_{t=0}^{p-1}\omega^{jt}(\sum_{l=1}^{p}\omega^{-lt}x_{il})$, we are done. $\square$

Lemma 3. *For all i, the irreducible character $\chi_i \in C_i$ is given by $\chi_i = (\sum_{j=1}^{p} x_{ij}) \times E_0$. In particular, $(I_B \otimes \varepsilon)(\chi_i) = \sum_{j=1}^{p} x_{ij}$ (where I_B is the identity map of B).*

Proof. Write $\chi_i = \sum_{j=1}^{p} x_{ij} \times h_j,\ h_j \in H$. Since $\chi_i g = \chi_i$, we get that for all j, $h_j g = h_j$ which implies that $h_j = \alpha_j E_0$ for some $\alpha_j \in k$. Since $g\chi_i = \chi_i$, we get, using Lemma 1, that $\alpha_j = \alpha$ for some $\alpha \in k$, and all j. The result follows now from the fact that $\varepsilon(\chi_i) = p$. $\square$

Recall that since A has an antipode S it follows that I_B has an inverse S_B under convolution, and that $S(b \times h) = \sum(1 \times S(b^{(1)}h))(S_B(b^{(2)}) \times 1)$ for all $h \in H$ and $b \in B$ [R]. Note that since x_{ij} is a grouplike element of B, it is invertible and $x_{ij}^{-1} = S_B(x_{ij})$.

Lemma 4. *For all i, $S(\chi_i) = (\sum_{j=1}^{p} x_{ij}^{-1}) \times E_0$. In particular, for all i there exists $k(i)$ such that $\sum_{j=1}^{p} x_{ij}^{-1} = \sum_{j=1}^{p} x_{k(i)j}$.*

Proof. By Lemma 2, $\rho(\sum_{j=1}^{p} x_{ij}) = 1 \otimes (\sum_{j=1}^{p} x_{ij})$, and thus the first claim follows by the formula for S. The second claim follows now from Lemma 3. $\square$

Lemma 5. *For any $1 \le i, k \le a,\ V_iV_k \in \{V_t | 1 \le t \le a\}$.*

Proof. Since V_iV_k is stable under the action of g, it is sufficient to show that it is a p-dimensional subcoalgebra of B. Indeed, using $\Delta_B(bb') = \sum b_{(1)}(b_{(2)}^{(1)} \cdot b'_{(1)}) \otimes b_{(2)}^{(2)} b'_{(2)}$ [R], it is straightforward to verify, using Lemma 2, that

$$\Delta_B(x_{ij}x_{kl}) = \sum_{t=1}^{p} x_{ij}(E_{j-t} \cdot x_{kl}) \otimes x_{it}x_{kl}.$$

Hence, V_iV_k is a subcoalgebra of B. Moreover, using Kaplansky's notation [K], $R(x_{i1}x_{k1}) = sp\{x_{it}x_{k1} | 1 \le t \le p\}$ has dimension p (since x_{k1} is invertible). But, by [K, Theorem 1] and the fact that B is commutative, $x_{ij}x_{kl} \in R(x_{i1}x_{kl}) = R(x_{kl}x_{i1}) \subseteq RR(x_{k1}x_{i1}) = R(x_{i1}x_{k1})$ for all $1 \le j, l \le p$ and the result follows. $\square$

We are ready now to compute $\chi_i S(\chi_i)$ in two ways and reach a contradiction. First, since $\sum_{j=1}^{p} x_{ij}$ and E_0 commute, it follows using Lemmas 4 and 5 that

$$\begin{aligned} \chi_i S(\chi_i) &= \left(\sum_{j=1}^{p} x_{ij}^{-1}\right)\left(\sum_{j=1}^{p} x_{ij}\right) \times E_0 \\ &= \left(\sum_{j=1}^{p} x_{k(i)j}\right)\left(\sum_{j=1}^{p} x_{ij}\right) \times E_0 = \left(\sum_{j=1}^{p} \alpha_j x_{mj}\right) \times E_0 \end{aligned} \tag{1}$$

for some scalars α_j and $1 \leq m \leq a$.

Second, let λ be an integral of A^* with $\varepsilon(\lambda) = 1$. Then by the orthogonality relation [L], $\chi_i S(\chi_i) = \sum_{j=0}^{p-1} \langle \chi_i S(\chi_i) g^j, \lambda \rangle g^j + \sum_{j=1}^{a} \langle \chi_i S(\chi_i)\chi_j, \lambda\rangle \chi_j$ and $\langle \chi_i S(\chi_i), \lambda \rangle = 1$. Since $\chi_i S(\chi_i) g^j = \chi_i S(\chi_i)$ for all j, counting dimensions yields that

$$\chi_i S(\chi_i) = \sum_{j=1}^{p} g^j + \sum_{j=1}^{p-1} \chi_{i_j} \tag{2}$$

for some irreducible characters χ_{i_j} of A^* of dimension p (not necessarily different).

Finally, applying $I_B \otimes \varepsilon$ to the right-hand sides of (1) and (2) yields that $\sum_{j=1}^{p} \alpha_j x_{mj} = p \cdot 1 + \sum_{j=1}^{p-1} \sum_{l=1}^{p} x_{i_j l}$ by Lemma 3. But this is a contradiction since $1 \notin \bigoplus_{i=1}^{a} V_i$. □

We remark that Y. Sommerhäuser has obtained a proof of Theorem 3.4(5) of [GW] by different methods.

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