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A NOTE ON DUALITY BETWEEN MEASURE AND CATEGORY

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ABSTRACT. We show that there is no Erdös–Sierpiński mapping preserving addition.

Let $\mathcal{M}$ and $\mathcal{N}$ be the ideals of meager and null subsets of 2^ω.

Definition 1. A bijection $F : 2^\omega \longrightarrow 2^\omega$ is called an Erdös–Sierpiński mapping if

$$X \in \mathcal{N} \iff F[X] \in \mathcal{M} \quad \text{and} \quad X \in \mathcal{M} \iff F[X] \in \mathcal{N}.$$

Theorem 2 ([4], [1]). *Assume* CH. *There exists an Erdös–Sierpiński mapping.* □

Since the existence of Erdös–Sierpiński mapping implies that the ideals $\mathcal{M}$ and $\mathcal{N}$ have the same cardinal characteristics, the existence of such mapping cannot be proved in ZFC.

Consider the space 2^ω as a topological group with addition modulo 2. The following question has been attributed to Erdös and Urbanik and to Ryll-Nardzewski:

Is it consistent that there is an Erdös–Sierpiński mapping F such that

$$\forall x, y \in 2^\omega \ F(x+y) = F(x) + F(y)?$$

The motivation for this question is the following (see [1] for more details):

Definition 3. Suppose that $X \subseteq 2^\omega$. We say that $X \in \mathcal{SN}$ (X has strong measure zero) if for every set $F \in \mathcal{M}$, $X + F \neq 2^\omega$.

$X \in \mathcal{SM}$ (X is strongly meager) if for every $H \in \mathcal{N}$, $X + H \neq 2^\omega$.

An Erdös–Sierpiński mapping satisfying $F(x+y) = F(x)+F(y)$ would also map strong measure zero sets onto strongly meager sets and vice versa.

Consider the following statement (considered by Carlson in [3]):

(φ) For every set $F \in \mathcal{M}$ there exists a set $F' \in \mathcal{M}$ such that

$$\forall x_1, x_2 \in 2^\omega \ \exists x \in 2^\omega \big((2^\omega \setminus F') + x_1\big) \cup \big((2^\omega \setminus F') + x_2\big) \subseteq (2^\omega \setminus F) + x.$$

Let $\varphi^\star$ be the dual statement obtained by replacing $\mathcal{M}$ with $\mathcal{N}$.

Note that φ implies that $\mathcal{SN}$ is an ideal and $\varphi^\star$ implies that $\mathcal{SM}$ is an ideal (see remarks at the end of the paper).

Theorem 4 (Carlson, [3]). ZFC $\vdash \varphi$.

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Proof. For completeness we present a short proof based on the following classical characterization of meager sets in 2^ω:

Theorem 5 ([1]). *A set $F \subseteq 2^\omega$ is meager if and only if there exists a partition of ω into intervals $\{I_n : n \in \omega\}$ and a function $x_F \in 2^\omega$ such that*

$$F \subseteq \{x \in 2^\omega : \forall^\infty n \; x \restriction I_n \neq x_F \restriction I_n\}. \quad \square$$

Suppose that $F \subseteq 2^\omega$ is a meager set. Without loss of generality we can assume that $F = \{x \in 2^\omega : \forall^\infty n \; x \restriction I_n \neq x_F \restriction I_n\}$ for some partition $\{I_n : n \in \omega\}$ and real $x_F \in 2^\omega$.

Let $J_n = I_{2n} \cup I_{2n+1}$ for every n. Define

$$F' = \{x \in 2^\omega : \forall^\infty n \; x \restriction J_n \neq 0\}.$$

Suppose that $x_1, x_2 \in 2^\omega$. Define $x = x_1 \restriction \bigcup_n I_{2n} \cup x_2 \restriction \bigcup_n I_{2n+1}$. It is clear that

$$\big((2^\omega \setminus F') + x_1\big) \cup \big((2^\omega \setminus F') + x_2\big) \subseteq (2^\omega \setminus F) + x.$$

A small modification of the above argument shows that we can consider more than two translations, countably many or even $< \mathsf{add}(\mathcal{N})$. $\square$

Theorem 6. $\mathsf{ZFC} \vdash \neg\varphi^\star$.

Proof. We start with the following easy observation:

Lemma 7. *Suppose that $I \subseteq \omega$ is a finite set and $J' \subseteq J \subseteq 2^I$ are such that*

1. *$|J'| \cdot 2^{-|I|} = 1 - \delta$ and $|J| \cdot 2^{-|I|} = 1 - \varepsilon$,*
2. *$\delta^2 < \varepsilon < \delta$.*

There exist $t_1, t_2 \in 2^I$ such that

$$\forall s \in 2^I \; (J' + t_1) \cup (J' + t_2) \not\subseteq J + s.$$

Proof. Let

$$Z = \{(t_1, t_2, z) : z \in (J' + t_1) \cup (J' + t_2)\}.$$

Check that for every $z \in 2^I$,

$$\frac{|2^I \times 2^I \setminus (Z)_z|}{2^{2\cdot|I|}} = \delta^2.$$

Thus $(Z)_z \cdot 2^{-2\cdot|I|} = 1 - \delta^2 > 1 - \varepsilon$ for all z. By the finitary version of Fubini's theorem there are t_1, t_2 such that

$$\frac{|(Z)_{t_1,t_2}|}{2^{|I|}} > 1 - \varepsilon.$$

In particular,

$$(Z)_{t_1,t_2} = (J' + t_1) \cup (J' + t_2) \not\subseteq J + s.$$

$\square$

Fix a partition of ω into finite sets $\{I_n : n \in \omega\}$ such that $|I_n| > 2^n$. For each n choose $J_n \subseteq 2^{I_n}$ such that

$$1 - \frac{1}{n^2} + \frac{1}{n^5} \geq \frac{|J_n|}{2^{|I_n|}} \geq 1 - \frac{1}{n^2}.$$

Let

$$F = \{x \in 2^\omega : \exists^\infty n \; x \restriction I_n \notin J_n\}.$$

The following lemma finishes the proof of Theorem 6.

Lemma 8. *For every null set $F' \supseteq F$ there are $x_1, x_2 \in 2^\omega$ such that for every $x \in 2^\omega$*

$$\Big((2^\omega \setminus F') + x_1\Big) \cup \Big((2^\omega \setminus F') + x_2\Big) \not\subseteq (2^\omega \setminus F) + x.$$

Proof. For a closed set $C \subseteq 2^\omega$, $n \in \omega$ and $s \in 2^{<\omega}$ let

$$C_s = \{x{\restriction}(|s|, \omega) : s \subseteq x\} \text{ and } C{\restriction}n = \{x{\restriction}n : x \in C\}.$$

Let

$$C = \{x \in 2^\omega : \forall n\ x{\restriction}I_n \in J_n\}.$$

Without loss of generality we can assume that C has positive measure. Suppose that F' is a null set. Let C' be a set of positive measure such that

$$F' \subseteq 2^\omega \setminus (C' + \mathbb{Q}),$$

where $\mathbb{Q} = \{x \in 2^\omega : \forall^\infty n\ x(n) = 0\}$.

We will construct two reals x_1, x_2 such that for every x

$$(C' + x_1) \cup (C' + x_2) \not\subseteq (2^\omega \setminus F) + x.$$

Define by induction an increasing sequence $\{n_k : k \in \omega\}$ and $x_i{\restriction}I_{n_k}$ for $i = 1, 2$. For $m \neq n_k$ we put $x_i{\restriction}I_m = 0$.

Suppose that $x_1{\restriction}I_1 \cup I_2 \cup \cdots \cup I_{n_k}$ and $x_2{\restriction}I_1 \cup I_2 \cup \cdots \cup I_{n_k}$ are defined. We need to define n_{k+1} and $x_i{\restriction}I_{n_{k+1}}$ for $i = 1, 2$. Use the Lebesgue density theorem to find sequences $\{r^s : s \in C'{\restriction}I_1 \cup I_2 \cup \cdots \cup I_{n_k}\}$ and $\ell > n_k$ such that

1. $\mathsf{dom}(s^\frown r^s) = I_1 \cup I_2 \cup \cdots \cup I_\ell$,
2. the set $\bigcap_s C'_{s^\frown r^s}$ has positive measure.

For $m \geq \ell$ let

$$J'_m = \left\{x{\restriction}I_m : x \in \bigcap_s C'_{s^\frown r^s}\right\}.$$

Note that

$$\bigcap_s C'_{s^\frown r^s} \subseteq \{x \in 2^\omega : \forall m\ x{\restriction}I_m \in J'_m\}.$$

Since the set on the left-hand side has positive measure there must be infinitely many m such that

$$\frac{|J'_m|}{2^{|I_m|}} > 1 - \frac{1}{m},$$

since $\prod_m \left(1 - \frac{1}{m}\right) = 0$. Let n_{k+1} be the first such m that is bigger than ℓ. Apply the lemma to get sequences t_1^{k+1}, t_2^{k+1} such that

$$\forall s \in 2^{I_{n_{k+1}}}\ (J'_{n_{k+1}} + t_1^{n_{k+1}}) \cup (J'_{n_{k+1}} + t_2^{n_{k+1}}) \not\subseteq J_{n_{k+1}} + s.$$

Define $x_1{\restriction}I_{n_{k+1}} = t_1^{k+1}$ and $x_2{\restriction}I_{n_{k+1}} = t_2^{k+1}$. This completes the definition of x_1 and x_2.

Suppose that $x \in 2^\omega$ is given. Let $s_n = x{\restriction}I_n$. Without loss of generality we can assume

$$\exists^\infty k\ (J'_{n_{k+1}} + t_1^{n_{k+1}}) \not\subseteq J_{n_{k+1}} + s_{n_{k+1}}.$$

Let $U \subseteq \omega$ be the set of k satisfying the requirement above. We will show that

$$C' + x_1 \nsubseteq (2^\omega \setminus F) + x.$$

For each k let $u_k \in J'_{n_k} + t_1^{n_k}$ be such that $u_k \in (J'_{n_k} + t_1^{n_k}) \setminus (J_{n_k} + s_{n_k})$ if possible, i.e. if $k \in U$.

Let $v_0 = r^\emptyset$ and

$$v_{k+1} = \begin{cases} v_k{}^\frown u_{k/2} & \text{if } k \text{ is even,} \\ v_k{}^\frown r^{v_k} & \text{if } k \text{ is odd.} \end{cases}$$

Let $z = \bigcup_k v_k$. Since $[v_k] \cap (C' + x_1) \neq \emptyset$ for all k, it follows that $z \in C' + x_1$. On the other hand $z \notin (2^\omega \setminus F) + x$ since

$$\exists^\infty k \; z \restriction I_{n_k} \notin J_{n_k} + s_{n_k}.$$

□

Remarks. 1. The proof shows that there is no Erdös–Sierpiński mapping F such that

$$\forall X \subseteq 2^\omega \; \forall y \in 2^\omega \; \exists z \in 2^\omega \; F[X+y] = F[X] + z.$$

2. It is consistent that $\mathcal{SM}$ is an ideal (of countable sets) ([3]). The Continuum Hypothesis implies that $\mathcal{SM}$ is not an ideal ([2]).

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