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EXISTENCE OF CRITICAL MODULES OF GK-DIMENSION 2 OVER ELLIPTIC ALGEBRAS

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Abstract. We show that over an elliptic algebra, critical modules of Gelfand-Kirillov dimension 2 exist in all multiplicities (assuming the ground field is uncountable, algebraically closed). Geometrically, this shows that in a quantum plane there exist "irreducible curve" modules of all possible degrees.

1. Introduction

The objective of this paper is to show that over an elliptic algebra A there exist critical modules of Gelfand-Kirillov (GK-) dimension 2 in *all* multiplicities. Interpreted geometrically in the language of non-commutative projective geometry, this means that in a quantum plane there exist "irreducible curve" modules of all possible degrees. It is well-known from [ATV2] that critical A-modules of GK-dimension 2 exist in multiplicity 1: up to a shift, these are precisely the line modules. But very little seems to be known about the critical modules of GK-dimension 2 in higher multiplicities; indeed, it is natural to question their existence. This paper proves their existence under the assumption that the ground field is uncountable, algebraically closed. The proof is geometric in that it uses the notion of the divisor of a module introduced in [Aj] and [AjV]. Along the way, we introduce the notion of a *quantum-irreducible divisor* (Definition 3.1), and establish a decomposition result (Theorem 3.4).

2. Preliminaries

The following conventions and notations will be used throughout the paper: A always denotes an elliptic Artin-Schelter regular algebra of dimension 3 with three generators, over an algebraically closed field k. Our main result (in the next section) requires k to be uncountable, but the results in this section do not. As in [ATV1], [ATV2], let $(E, \sigma, \mathcal{L})$ be the triplet associated to the algebra A. We assume throughout that σ has infinite order and no fixed point. All modules are assumed to be graded *right* modules, and homomorphisms between them degree-preserving. There is a normalizing element g of degree 3 (unique up to a scalar) in

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A. For an A-module M, we define the g-torsion submodule M_g of M as the graded A-submodule whose elements are annihilated by a power of g. We say that M is *g-torsion* (resp. *g-torsion-free*) if $M_g = M$ (resp. $M_g = 0$).

For definitions and basic results on modules over A, we refer to [ATV2], [Aj]. Recall [Aj], in particular, that an A-module M is called *normal* if M is Cohen-Macaulay and its Hilbert function h_M satisfies the condition: $h_M(n) = 0$ if and only if $n < 0$. It is worth pointing out that the definition of a *normal* A-module given here is not the same as (but, is more general than) that of a *normalized* A-module sometimes used in the literature. In [ATV2, §6] an A-module of GK-dimension 1 is defined to be normalized if it is Cohen-Macaulay and has the Hilbert series $e/(1-t)$ for some natural number e. A direct generalization of this would lead us to define an A-module of GK-dimension 2 to be normalized if it is Cohen-Macaulay and has the Hilbert series $(e_1 + e_2 t)/(1-t)^2$ for some natural numbers e_1 and e_2. This definition of normalization completely specifies the Hilbert function, rather than just requiring the Hilbert function to be zero in negative degrees. Of course, every normalized module is normal, but the class of normal modules is much bigger than that of normalized ones: indeed, every Cohen-Macaulay module is a shift of a normal module but not necessarily a shift of a normalized one. For example, $N \oplus N'(-n)$, where N is a point module and $n > 0$, is normal but not normalized.

We will use the following definition throughout the paper: a **curve A-module** is a g-torsion-free A-module of GK-dimension 2. Of course, when M has some additional property such as being normal (resp. critical) A-module, we call M a **normal curve A-module** (resp. **critical curve A-module**). The motivation for the term curve modules comes from their geometric interpretation in terms of non-commutative projective schemes: curve A-modules correspond to curves in the quantum plane $\mathrm{proj} - A$. At first glance it might seem more natural to define a curve A-module to be just any A-module of GK-dimension 2, but the following explains why we require a curve module to be g-torsion-free. Crucial to our analysis is the notion of a divisor $\mathrm{Div}(M)$ on E, associated to a curve module M, which describes the points of the curve represented by M in the quantum plane (the points in the quantum plane being identified with those on E). In the next paragraph, $\mathrm{Div}(M)$ is defined for g-torsion-free A-modules M of GK-dimension 2. Now, had we defined a curve module to be just any A-module of GK-dimension 2, we would have to deal with the unpleasant situation that the notion of such a divisor fails for g-torsion modules of GK-dimension 2 because they "vanish" at all points of E. More precisely, if M is a g-torsion A-module of GK-dimension 2, then M maps to all point modules (and, a curve in the quantum plane should not pass through all the points in the plane!).

Now we recall from [AjV], [Aj] the notion of the divisor associated to a curve A-module, and give some more basic facts to be used in the proof of the main result. Since the ring $B = A/(g)$ is isomorphic to the twisted coordinate ring of the triplet $(E, \sigma, \mathcal{L})$ [ATV1], [ATV2], it follows by [AV] that there is a category equivalence $(gr - B)/(gr - B)_0 \longrightarrow \mathrm{coh}(E) : X \mapsto \tilde{X}$, reducing the GK-dimension by one, where $\mathrm{coh}(E)$ denotes the category of coherent $\mathcal{O}_E$-modules. For a curve A-module M, M/gM is a B-module of GK-dimension 1, so $\widetilde{M/gM}$ is a finite dimensional $\mathcal{O}_E$-module which corresponds to a divisor on E. We call this divisor *the divisor of M* and denote it by $\mathrm{Div}(M)$. The following elementary facts about $\mathrm{Div}(M)$ are immediate [AjV].

Proposition 2.1. *Let M be a curve A-module.*

(i) The degree of $\mathrm{Div}(M)$ *is* $3d$ *where d is the multiplicity of M.*

(ii) For any integer n, $\mathrm{Div}(M(n)) = \sigma^n(\mathrm{Div}(M))$.

(iii) Div *is additive on exact sequences: For a short exact sequence* $0 \to M' \to M \to M'' \to 0$ *of curve A-modules,*

$$\mathrm{Div}(M) = \mathrm{Div}(M') + \mathrm{Div}(M''). \tag{2.2}$$

When M is a *normal* curve A-module, there is another divisor $\mathrm{div}(M)$, defined in [Aj] as the divisor of zeroes $\mathrm{div}(s_M)$ of a certain global section s_M of a certain invertible sheaf on E. To make this precise, recall that normal A-modules of GK-dimension 2 can be classified into different *types* [Aj] according to the forms of their minimal projective resolution. Having projective dimension 1, such a module has a minimal projective resolution (unique up to isomorphism) of the form

$$0 \longrightarrow A(-j_1) \oplus \ldots \oplus A(-j_r) \to^{\phi_M} A(-i_1) \oplus \ldots \oplus A(-i_r) \longrightarrow M \to 0, \tag{2.3}$$

where the indices i_ν, j_ν indicate the shifts of the module. The sequence of indices $\tau = (i_1, \ldots, i_r; j_1, \ldots, j_r)$, which is unique if we arrange the sequences $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$ in non-decreasing order, is called the *type* of M. The map ϕ_M is given by a square matrix of homogeneous elements of A. Now, as shown in [Aj, Theorem 3.26], using these homogeneous elements in an appropriate way one can define a global section s_M of an invertible sheaf $\mathcal{L}_\tau$ on E, where s_M is independent (up to a scalar multiple) of the choice of minimal resolution for M and $\mathcal{L}_\tau$ depends only on the type τ of M. Explicit formulas for s_M and $\mathcal{L}_\tau$ are given in [Aj, 3.23 and 3.20] respectively. In fact, s_M is the determinant of a matrix obtained by appropriately "σ-twisting " the matrix representing the map ϕ_M; and it is easy to write down a formula for $\mathcal{L}_\tau$: For $\tau = (i_1, \ldots, i_r; j_1, \ldots, j_r)$, the expression

$$p_\tau(\sigma) = \frac{\sum_{k=1}^{r} (\sigma^{j_k} - \sigma^{i_k})}{(\sigma - 1)}$$

is a polynomial in σ. Then

$$\mathcal{L}_\tau = \mathcal{L}^{p_\tau(\sigma)}, \tag{2.4}$$

where the right hand side means the action of $p_\tau(\sigma)$ on $\mathcal{L}$, considering $Pic(E)$ as a module over $\mathbb{Z}[\sigma]$ (the action of σ being the pull-back by the functor σ^*). It follows from M being g-torsion-free that s_M is not the identically zero section. Now $\mathrm{div}(M)$ is defined to be the divisor of zeroes $\mathrm{div}(s_M)$ of the section s_M.

A key observation which follows immediately from above is that the linear equivalence class of the divisor $\mathrm{div}(M)$ depends only on the form of the minimal projective resolution of M. More precisely,

Lemma 2.5. *If M and M' are curve A-modules of the same type, then* $\mathrm{div}(M)$ *and* $\mathrm{div}(M')$ *are linearly equivalent divisors on E.*

It is easy to see the relation between the two divisors $\mathrm{Div}(M)$ and $\mathrm{div}(M)$.

- First, $\mathrm{Div}(M)$ coincides with $\mathrm{div}(M)$ when M is a normal curve A-module.
- Furthermore, for an arbitrary curve A-module M,

$$\mathrm{Div}(M) = \sigma^n \,\mathrm{div}(M'),$$

where M' is normal and n is an integer.

To see the last fact, recall the canonical map $\mu_M : M \to M^{\vee\vee}$ [ATV2], whose kernel has GK-dimension ≤ 1 and cokernel is finite k-dimensional. Since M is g-torsion-free and every A-module of GK-dimension ≤ 1 is g-torsion, the kernel has to vanish. Thus $\mathrm{Div}(M) = \mathrm{Div}(M^{\vee\vee})$, but $M^{\vee\vee} = M'(n)$ for some normal M'.

Definition 2.6. Let D be an effective divisor on E. We say that a point $q \in E$ is a **supplement of** D if

$$D + q = \mathrm{div}(M)$$

for some critical normal curve A-module M.

Needless to say, in view of 2.1(i), an effective divisor D on E has a supplement only if $\deg(D) = 3d - 1$ for some positive integer d.

Lemma 2.7. *Let D be an effective divisor on E. The set of supplements of D is finite (possibly empty).*

Proof. Assume that the set of supplements of D is non-empty, and let $\deg(D) = 3d - 1$. Suppose $D + q = \mathrm{div}(M)$, $D + q' = \mathrm{div}(M')$, where M, M' are critical normal curve A-modules of multiplicity d. If M and M' are of the same type, then by Lemma 2.5 $D+q$ and $D+q'$ are linearly equivalent, hence $q = q'$. Thus we have shown that for a fixed type τ, the set of points $q \in E$ such that $D+q = \mathrm{div}(M)$ for some critical normal M of type τ, has at most one element. Since there are only finitely many types of critical normal A-modules of GK-dimension 2 and multiplicity d [Aj, 3.8], the result follows. □

Remark 2.8. Let ϕ be a homogeneous element of A of degree $d > 0$, and let $\overline{\phi}$ denote the image of ϕ in $B = A/(g)$, which we regard as a global section of the invertible sheaf $\mathcal{L}_d = \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{d-1}}$ on E. Then [Aj, 3.30], $\mathrm{div}(A/\phi A) = \mathrm{div}(\overline{\phi})$, so a point $p \in E$ is in the support of $\mathrm{div}(A/\phi A)$ if and only if $\overline{\phi}(p) = 0$. Now, since the space $H^0(E, \mathcal{L}_d)$ has dimension $3d$, it is possible to find a homogeneous element $\phi \in A$ of degree d such that $\overline{\phi}$ vanishes at any given $3d - 1$ distinct points of E. Thus, given a multiplicity-free effective divisor D of degree $3d - 1$, there exists a homogeneous element $\phi \in A$ of degree d such that $\mathrm{div}(A/\phi A) = D + q$ for some point $q \in E$. This remark will provide a key to our existence theorem.

3. Existence of critical modules

In this section, we first define the notion of quantum-irreducibility of an effective divisor on E, and observe that quantum-irreducibility of $\mathrm{Div}(M)$ implies criticality of M. Then we prove the existence of critical modules by constructing appropriate quantum-irreducible divisors.

Definition 3.1. Let D be an effective divisor on E. We say that D is **quantum-reducible** if

$$D = \mathrm{Div}(M) + D', \tag{3.2}$$

where M is a curve A-module and D' is an effective divisor of positive degree. We say D is **quantum-irreducible** if it is not quantum-reducible.

Note that in (3.2), the multiplicity of M is less than $\deg(D)/3$. Thus, any effective divisor of degree 1,2, or 3 is trivially quantum-irreducible.

Lemma 3.3. *Let M be a curve A-module. If* $\operatorname{Div}(M)$ *is quantum-irreducible, then M is critical. More precisely, if M is not critical, then*

$$\operatorname{Div}(M) = \operatorname{Div}(M_1) + \operatorname{Div}(M_2) \tag{3.4}$$

for some curve A-modules M_1 and M_2, where M_2 is critical.

Proof. If M is not critical, then it has a critical quotient M_2 of GK-dimension 2, so there exists an exact sequence $0 \to M_1 \to M \to M_2 \to 0$. Since M is g-torsion-free, M_1 is too, and $\operatorname{GKdim}(M_1) = 2$. Now M_2 is also g-torsion-free for the following reason. A critical A-module is either g-torsion-free or is a B-module. If M_2 were a B-module, then the above exact sequence tensored with B would yield an exact sequence $M/gM \longrightarrow M_2 \to 0$, implying $2 = \operatorname{GKdim}(M_2) \leq \operatorname{GKdim}(M/gM) = 1$. So M_2 must be g-torsion-free. Now the result follows by applying Lemma 2.1(iii) to the above exact sequence. □

A repeated application of the above lemma gives the following decomposition of $\operatorname{Div}(M)$.

Theorem 3.5. *Let M be a curve A-module. Then*

$$\operatorname{Div}(M) = \sum_i \operatorname{Div}(M_i) = \sum_i \sigma^{n_i} \operatorname{div}(M_i'),$$

where M_i (resp. M_i') are critical (resp. critical normal) curve A-modules. □

Note that this is a rather weak statement, because it is not known whether the divisor of a critical module is necessarily quantum-irreducibile. For the conic modules, Theorem 3.5 takes the following form, making [Aj, Proposition 4.9, and the remark following that] more precise.

Corollary 3.6. *Let M be a conic A-module, i.e., a curve A-module of multiplicity 2. If M is not critical, then*

$$\operatorname{Div}(M) = \sigma^{n_1} \operatorname{div}(M_{l_1}) + \sigma^{n_2} \operatorname{div}(M_{l_2}),$$

where M_{l_1} and M_{l_2} are line modules. □

We now show how to construct quantum-irreducible divisors. It is useful to introduce the following definition: an effective divisor D on E is called **a divisor of a quantum curve** if $D = \operatorname{Div}(M)$ for some curve A-module M. Trivially, for D to be a divisor of a quantum curve, D must have degree $3d$ for some positive integer d.

Theorem 3.7. *Assume that the ground field is uncountable. For every positive integer N, there exists a multiplicity-free quantum-irreducible effective divisor D on E of degree N, which is* not *a divisor of a quantum curve.*

Proof. We use induction on N. The cases $N = 1, 2$ are trivial. Assume that we have a multiplicity-free quantum-irreducible effective divisor D of degree N, which is not a divisor of a quantum curve.

Consider the set Ω of all possible effective divisors, whose support is contained in $\operatorname{supp}(D)$ and whose degree is at most N. The set Ω is clearly finite. The set

$$\Omega' = \{\, \sigma^n D_1 \mid n \in \mathbb{Z}, D_1 \in \Omega \,\}$$

is countable. By Lemma 2.7, the set of supplements of a fixed $\Delta \in \Omega'$ is finite. Let Γ denote the union of the sets of supplements of various Δ, the union being taken over $\Delta \in \Omega'$. The set Γ is countable. Let X be the union of the orbits of various points p, the union being taken over $p \in \Gamma \cup \mathrm{supp}(D)$. The set X is also countable. Now since the ground field k is uncountable algebraically closed, the set of k-points of E is uncountable (indeed, the cardinality of any curve is the cardinality of the ground field). So, $E - X$ is non-empty. Choose any point $q \in E - X$. We claim that $D + q$ is a multiplicity-free quantum-irreducible divisor of degree $N + 1$, which is not a divisor of a quantum curve.

Clearly, $D + q$ is multiplicity-free. We prove that $D + q$ is quantum-irreducible. If not, then the definition of quantum-reducibility and Theorem (3.5) imply that

$$D + q = \sigma^n \operatorname{div}(M) + D_1 \tag{3.8}$$

for some critical normal curve A-module M, and some effective divisor D_1 of positive degree. Suppose that $q \in \mathrm{supp}(D_1)$. Then $D = \sigma^n \operatorname{div}(M) + (D_1 - q)$. Since D is not a divisor of a quantum curve, $D_1 - q \neq 0$. But this implies that D is quantum-reducible, which is a contradiction. Therefore, $q \notin \mathrm{supp}(D_1)$. Then, by (3.8), $\sigma^n \operatorname{div}(M) = q + D_2$ for some $D_2 \in \Omega$. This implies that $\sigma^{-n} q$ is a supplement of $\sigma^{-n} D_2$, which contradicts the choice $q \in E - X$. This proves the quantum-irreducibility of $D + q$.

We now show that $D + q$ is not a divisor of a quantum curve. If it were, then $D + q = \operatorname{Div}(M) = \sigma^n \operatorname{div}(M')$ for some normal curve A-module M'. Since we already know that $D+q$ is quantum-irreducible, it follows that M is critical (Lemma 3.3), and so is M'. But then this means that $\sigma^{-n} q$ is a supplement of $\sigma^{-n} D$, contradicting the choice $q \in E - X$. □

Theorem 3.9. *Assume that the ground field is uncountable. For every positive integer d, there exists a critical normal curve A-module of multiplicity d. More precisely, for every positive integer d, there exists a homogeneous element $\phi \in A$ of degree d such that the A-module $A/\phi A$ is a g-torsion-free critical normal A-module of GK-dimension 2 and multiplicity d.*

Proof. By Theorem 3.7, there exists a multiplicity-free quantum-irreducible effective divisor D of degree $3d-1$. By Remark 2.8, there exists a homogeneous element $\phi \in A$ of degree d and a point $q \in E$ such that $D + q = \operatorname{div}(A/\phi A)$. If $A/\phi A$ were not critical, then by Lemma 3.3, $D+q = \operatorname{div}(A/\phi A) = \operatorname{Div}(M_1) + \operatorname{Div}(M_2)$ for some curve A-modules M_1 and M_2. This would imply that at least one of the two divisors $(\operatorname{Div}(M_1) - q)$ and $(\operatorname{Div}(M_2) - q)$ is effective of positive degree. Then at least one of the two equalities $D = \operatorname{Div}(M_1) + (\operatorname{Div}(M_2) - q)$ and $D = (\operatorname{Div}(M_1) - q) + \operatorname{Div}(M_2)$ will contradict the quantum-irreducibility of D. □

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References

[Aj] K. Ajitabh, *Modules over elliptic algebras and quantum planes*, Proc. Lond. Math. Soc. (3)**72** (1996), 567-587. MR **97a:**16049

[AjV] K. Ajitabh and M.Van den Bergh, *Presentation of critical modules of GK-dimension 2 over elliptic algebras*, Proc. American Math. Soc. **127**, Number 6 (1999), 1633–1639. MR **99i:**16046

[ATV1] M. Artin, J. Tate, and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift I, 33-85, Birkhäuser, Boston, 1990. MR **92e:**14002

[ATV2] M. Artin, J. Tate, and M. Van den Bergh, *Modules over regular algebras of dimension 3*, Invent. Math. **106** (1991), 335-388. MR **93e:**16055

[AV] M. Artin and M. Van den Bergh, *Twisted homogeneous coordinate rings*, J. Algebra **133** (1990), 249-271. MR **91k:**14003

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