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A FINITENESS RESULT FOR ASSOCIATED PRIMES OF LOCAL COHOMOLOGY MODULES

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ABSTRACT. We show that the first non-finitely generated local cohomology module $H^i_{\mathfrak{a}}(M)$ of a finitely generated module M over a noetherian ring R with respect to an ideal $\mathfrak{a} \subseteq R$ has only finitely many associated primes.

1. INTRODUCTION

Apparently very little is known about the finiteness of the set $\operatorname{Ass}_R\big(H^i_{\mathfrak{a}}(M)\big)$ of associated primes of the local cohomology module $H^i_{\mathfrak{a}}(M)$ of a finitely generated module M over a noetherian ring R with respect to an ideal $\mathfrak{a}$ of R. So, in [M-S] the authors ask whether the sets $\operatorname{Ass}_R\big(\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)\big)$ become stable for $n >> 0$. An affirmative answer to this would imply that the set $\operatorname{Ass}_R\big(H^i_{\mathfrak{a}}(M)\big)$ is always finite. On the other hand it seems not to be known whether the set $\operatorname{Ass}_R\big(H^2_{\mathfrak{a}}(M)\big)$ is finite in general even if $\mathfrak{a}$ is generated by two elements only. Moreover, if for two elements $a, b \in R$ the set $S_M(a,b) := \bigcup_{n\in\mathbb{N}} \operatorname{Ass}_R\ \big(M/(a^n, b^n)M\big)$ is finite, then $\operatorname{Ass}_R\big(H^2_{(a,b)}(M)\big)$ is finite. But Katzman [K] has shown that $S_M(a,b)$ need not be finite. In [B-R-Sh] we have shown that $S_M(a,b)$ is finite under certain conditions. These conditions imply that $H^1_{(a,b)}(M)$ is finitely generated and so the resulting finiteness of $\operatorname{Ass}_R\big(H^2_{(a,b)}(M)\big)$ is not surprising, as $\operatorname{Ass}_R\big(H^2_{\mathfrak{a}}(M)\big)$ is finite whenever $H^1_{\mathfrak{a}}(M)$ is finitely generated (see [B-R-Sh, (2.4), (2.5)]).

Finally, let us mention that the finiteness of the sets $\operatorname{Ass}_R\big(H^i_{\mathfrak{a}}(M)\big)$ is related to the *local-global-principle for finiteness dimensions* due to Faltings [F] (cf. [B-Sh, (9.6.2)]) and also to the open problem of whether such a principle holds for the annihilation of local cohomology modules (see [R], [B-R-Sh]).

The aim of this note is to show that the set $\operatorname{Ass}_R\big(H^i_{\mathfrak{a}}(M)\big)$ is finite, whenever the modules $H^1_{\mathfrak{a}}(M), \ldots, H^{i-1}_{\mathfrak{a}}(M)$ are finitely generated. This generalizes the corresponding result which is shown in [B-R-Sh] for the special case $i \leq 2$ and which was already mentioned above.

Throughout this note, let R be a noetherian ring, let $\mathfrak{a} \subseteq R$ be an ideal and let M be a finitely generated R- module. If $i \in \mathbb{N}_0$, we write $H^i_{\mathfrak{a}}(M)$ for the i-th local cohomology module of M with respect to the ideal $\mathfrak{a}$. For convenience we write

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$H^j_{\mathfrak{a}}(M) = 0$, whenever j is a negative integer. For the unexplained terminology we refer to [B-Sh].

2. The finiteness results

Proposition 2.1. *Let $i \in \mathbb{N}_0$ be such that $H^j_{\mathfrak{a}}(M)$ is finitely generated for all $j < i$ and let $N \subseteq H^i_{\mathfrak{a}}(M)$ be a finitely generated submodule. Then, the set $Ass_R\big(H^i_{\mathfrak{a}}(M)/N\big)$ is finite.*

Proof. We proceed by induction on i. The case $i = 0$ is obvious as $H^0_{\mathfrak{a}}(M)$ is finitely generated. So, let $i > 0$ and set $\overline{M} := M/\Gamma_{\mathfrak{a}}(M)$. As $H^0_{\mathfrak{a}}(\overline{M}) = 0$ and in view of the natural isomorphisms $H^k_{\mathfrak{a}}(\overline{M}) \cong H^k_{\mathfrak{a}}(M)$ for all $k \in \mathbb{N}$, we may replace M by $\overline{M}$ and hence assume that $\Gamma_{\mathfrak{a}}(M) = 0$. We thus find an M-regular element $y \in \mathfrak{a}$. By our choice of N, there is some $n \in \mathbb{N}$ with $y^n N = 0$.

We set $x := y^n$ and apply cohomology to the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. It follows that $H^l_{\mathfrak{a}}(M/xM)$ is finitely generated for all $l < i - 1$. Moreover, we get the following commutative diagram with exact rows and columns in which δ is the connecting homomorphism and in which ε and ϱ are the natural maps:

$(*)$

$$\begin{array}{ccccccc}
H^{i-1}_{\mathfrak{a}}(M) & \xrightarrow{\varepsilon} & H^{i-1}_{\mathfrak{a}}(M/xM) & \xrightarrow{\delta} & H^i_{\mathfrak{a}}(M) & \xrightarrow{x} & H^i_{\mathfrak{a}}(M) \\
 & & \downarrow & & \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \|} \\
0 & \longrightarrow & H^{i-1}_{\mathfrak{a}}(M/xM)/\delta^{-1}(N) & \xrightarrow{\overline{\delta}} & H^i_{\mathfrak{a}}(M)/N & \xrightarrow{\overline{x}} & H^i_{\mathfrak{a}}(M) \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

As $\mathrm{Ker}(\delta) = \varepsilon(H^{i-1}_{\mathfrak{a}}(M))$ and N are both finitely generated, so is $\delta^{-1}(N)$. Therefore, by induction

$$T := H^{i-1}_{\mathfrak{a}}(M/xM)/\delta^{-1}(N)$$

has only finitely many associated primes. It thus suffices to show the inclusion

$$(**) \qquad \mathrm{Ass}_R\big(H^i_{\mathfrak{a}}(M)/N\big) \subseteq \mathrm{Ass}_R(T) \cup \mathrm{Ass}_R(N) \ .$$

So, let $\mathfrak{p} \in \mathrm{Ass}_R\big(H^i_{\mathfrak{a}}(M)/N\big) \setminus \mathrm{Ass}_R(T)$. With an appropriate $h \in H^i_{\mathfrak{a}}(M)$ we may write $\mathfrak{p} = N \underset{R}{:} h$, hence $\mathfrak{p} = 0 \underset{R}{:} \varrho(h)$. As $\mathfrak{p} \notin \mathrm{Ass}_R(T)$, the last equality and the second row of $(*)$ show that $\mathfrak{p} \in \mathrm{Ass}_R\big(\overline{x}(\varrho(h))R\big) = \mathrm{Ass}_R(xhR)$. This allows us to write $\mathfrak{p} = 0 \underset{R}{:} xsh$ for some $s \in R$.

As xsh is annihilated by some power of x, we have $x \in \mathfrak{p}$. By our choice of h this means that $xsh \in N$. This implies that $\mathfrak{p} \in \mathrm{Ass}_R(N)$ and hence proves the inclusion $(**)$ and thus our result. □

Now, the announced result follows easily.

Theorem 2.2. *Let $i \in \mathbb{N}_0$ be such that $H^j_{\mathfrak{a}}(M)$ is finitely generated for all $j < i$. Then the set $Ass_R\big(H^i_{\mathfrak{a}}(M)\big)$ is finite.*

Proof. Apply Proposition (2.1) with $N = 0$. □

Next, let us introduce the $\mathfrak{a}$-*finiteness dimension* of M (see [B-Sh, (9.1.3)]):

$$f_{\mathfrak{a}}(M) := \min\{j \in \mathbb{N}_0 \mid H^j_{\mathfrak{a}}(M) \quad \text{not finitely generated}\} .$$

Using this notation we may write Theorem (2.2) in the form

Corollary 2.3. *If $i \leq f_{\mathfrak{a}}(M)$, then $Ass_R\big(H^i_{\mathfrak{a}}(M)\big)$ is a finite set.* □

Corollary 2.4 (see [B-R-Sh, (2.2)]). *The set $Ass_R\big(H_{\mathfrak{a}}^{grade_M(\mathfrak{a})}(M)\big)$ is finite.*

Proof. Follows from Corollary (2.3) as $\text{grade}_M(\mathfrak{a}) \leq f_{\mathfrak{a}}(M)$. □

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