

ON REFLEXIVITY OF DIRECT SUMS

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ABSTRACT. Necessary and sufficient conditions are presented to insure that the direct sum of two reflexive representations of a finite dimensional algebra is reflexive, and it is shown that for each such algebra, there is an integer k such that the direct sum of k copies of each of its representations is reflexive. Given a ring Δ , our results are actually presented in the more general setting of Δ -representations of a ring R .

Over the last thirty years, a significant amount of attention has been given to the problem of determining when an algebra A of operators on a vector space V (often a Hilbert space) over a field K is *reflexive* in the sense that no larger algebra of operators on that space has the same lattice of invariant subspaces. (See, for example, [2], [3], [6], [7], [12], [13], [14].) Of course, A is an algebra of operators on a vector space V if and only if V is a faithful left A -module. Thus, to attack this problem from a more module theoretic point of view, based on notation of Halmos [14], the following notions were presented in [9]: For any $R - \Delta$ -bimodule $M = {}_R M_\Delta$ one defines

$$\text{alglat}(M) = \{\alpha \in \text{End}(M_\Delta) \mid \alpha L \subseteq L \text{ for all } {}_R L \leq {}_R M\},$$

and, letting $\lambda : R \rightarrow \text{alglat}(M_\Delta)$ denote the canonical ring homomorphism, M is called a *reflexive* bimodule (or Δ -representation of R) if λ is surjective. Thus the K -algebra of operators A on V is reflexive if and only if ${}_A V_K$ is a reflexive bimodule (and then we simply say that the A -module V is reflexive).

The problems we shall deal with here have their roots in the papers [5] and [6] of Deddens and Fillmore and [2] of Azoff. Stated in module theoretic terms, in the first pair of papers the question was posed and answered (affirmatively) of whether a direct sum of two finitely generated reflexive modules over an algebra generated by a single complex matrix is again reflexive; and Azoff showed that for each finite dimensional module over a \mathbb{C} -algebra, there is a positive integer k , depending on its dimension, such that the direct sum of k copies of that module is reflexive. Other results in this vein can be found in [4] where Brenner and Butler showed that the direct sum of two copies of the regular representation of a finite dimensional algebra is reflexive, and in Habibi and Gustafson's [12] from which the same result follows for any faithful representation of a split serial algebra. (See [9], [10] and [11] for related and more general results.)

Our main objectives are to provide, in Theorem 1.2, a necessary and sufficient condition for a direct sum of two reflexive bimodules to be reflexive; and to show, in

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Theorem 3.2, that whenever R is a left artinian ring with composition length $c({}_R R)$, there is a positive integer $k \leq c({}_R R) + 1$ such that the direct sum $M^{(k)}$ of k copies of M is reflexive for any bimodule ${}_R M_\Delta$. Along the way we show how Theorem 1.2 can be employed to obtain simple proofs of some known results, and we examine a common generalization of generating and cogenerating called controlling that was introduced and employed in [9] and [10], showing in particular that if ${}_R M$ is faithful and reflexive, then $M \oplus N$ is reflexive if and only if M controls N .

1. A CHARACTERIZATION

Unless otherwise specified, all modules under consideration will be left- R , right- Δ bimodules for a fixed pair of rings R and Δ . If R is an algebra over a field K , we shall assume that $\Delta = K$. If $\alpha \in \text{End}(M_\Delta)$ and $\beta \in \text{End}(N_\Delta)$, we shall write $(\alpha, \beta) \in \text{End}(M \oplus N)$ for the direct sum map $(\alpha, \beta) : (m, n) \mapsto (\alpha m, \beta n)$. Thus if $\gamma \in \text{alglat}(M \oplus N)$, then $\gamma = (\gamma|_M, \gamma|_N)$.

A *common subquotient* of a pair of left modules ${}_R M$ and ${}_R N$ is a module ${}_R X$ that is isomorphic to a subquotient of both ${}_R M$ and ${}_R N$. This notion together with the following lemma allows us to determine just when the direct sum of a pair of reflexive modules is reflexive.

Lemma 1.1. *If $r, s \in R$, the ordered pair $(r, s) \in \text{alglat}(M \oplus N)$ if and only if $(r - s)X = 0$, for every (equivalently, every cyclic) common subquotient of ${}_R M$ and ${}_R N$.*

Proof. Note that $a \in R$ annihilates every common subquotient of M and N if and only if it annihilates each of their cyclic common subquotients. According to Gouratsats' Lemma, W is an R -submodule of $M \oplus N$, if and only if there are submodules

$$M_2 \leq M_1 \leq M \text{ and } N_2 \leq N_1 \leq N$$

and an R -isomorphism

$$f : M_1/M_2 \rightarrow N_1/N_2$$

such that

$$W = \{(m_1, n_1) \in M_1 \oplus N_1 \mid f(m_1 + M_2) = n_1 + N_2\}.$$

(Given W and the orthogonal projections π_M and π_N for $M \oplus N$, one checks that $M_1 = \pi_M(W)$, $M_2 = M \cap W$, $N_1 = \pi_N(W)$, and $N_2 = N \cap W$.)

(\Rightarrow) If $(m_1, n_1) \in W$ and $(rm_1, sn_1) \in W$, then

$$f(rm_1 + M_2) = sn_1 + N_2 = sf(m_1 + M_2)$$

so, since f is an R -isomorphism, $(r - s)(M_1/M_2) = 0$.

(\Leftarrow) Suppose $(r - s)(M_1/M_2) = 0$, and $f : M_1/M_2 \rightarrow N_1/N_2$ is an isomorphism and $(m_1, n_1) \in W$. Then

$$f(rm_1 + M_2) = rf(m_1 + M_2) = rn_1 + N_2 = sn_1 + N_2$$

so $(rm_1, sn_1) \in W$. □

Now we are able to provide the promised characterization in terms of annihilators of subquotients of M and N . The left *annihilator* of a module M is $\ell_R(M) = \{r \in R \mid rM = 0\}$.

Theorem 1.2. *Let M and N be reflexive, and let $\{X_i \mid i \in I\}$ represent one copy of each of the (cyclic) common subquotients of ${}_R M$ and ${}_R N$. Then $M \oplus N$ is reflexive if and only if*

$$\ell_R(M) + \ell_R(N) = \ell_R\left(\bigoplus_{i \in I} X_i\right).$$

Proof. (\Leftarrow) Let $\alpha = (\beta, \gamma) \in \text{alglat}(M \oplus N)$. Then there are $r, s \in R$ with $\beta = r$ and $\gamma = s$, and by Lemma 1.1, $r - s \in \ell_R(\bigoplus_{i \in I} X_i)$. But then by hypothesis

$$r - s = p - q \quad \text{with } p \in \ell_R(M) \text{ and } q \in \ell_R(N)$$

so that, letting

$$t = r - p = s - q,$$

we have $\alpha = \lambda(t)$.

(\Rightarrow) Let $r \in \ell_R(\bigoplus_{i \in I} X_i)$. Then by Lemma 1.1, $(r, 0) \in \text{alglat}(M \oplus N)$. Thus, assuming that $M \oplus N$ is reflexive,

$$(r, 0) = (s, s)$$

and

$$r = (r - s) + s \in \ell_R(M) + \ell_R(N).$$

Since always $\ell_R(M) + \ell_R(N) \subseteq \ell_R(\bigoplus_{i \in I} X_i)$, the proof is complete. \square

We note that the proof (\Rightarrow) above shows that if $M \oplus N$ is reflexive (regardless of reflexivity of M and N), then $\ell_R(M) + \ell_R(N) = \ell_R(\bigoplus_{i \in I} X_i)$.

From the inclusions

$$0 \subseteq \ell_R(M) + \ell_R(N) \subseteq \ell_R\left(\bigoplus_{i \in I} X_i\right) \subseteq \ell_R(X_i) \subseteq R$$

we easily obtain the following two corollaries:

Corollary 1.3. *If M and N are reflexive and have a common faithful subquotient, then $M \oplus N$ is reflexive.*

Corollary 1.4. *If R is semiperfect and M and N are reflexive and have no common composition factor, then $M \oplus N$ is reflexive.*

Proof. Suppose that $e_1, \dots, e_m, e_{m+1}, \dots, e_n$ is a complete set of primitive idempotents such that Re_i/Je_i is not a composition factor of M for $i = 1, \dots, m$, and Re_i/Je_i is not a composition factor of N for $i = m+1, \dots, n$. Then $\ell_R(M) + \ell_R(N) = R$. (See [1, Section 27].) \square

Regarding the Deddens-Fillmore result, the algebra generated by a matrix over a field K , being isomorphic to a proper factor of the polynomial ring $K[x]$, is an example of a split commutative uniserial algebra, i.e., a direct product of local uniserial rings. Since a module over a direct product of rings is reflexive precisely when its corresponding components are reflexive, to show that direct sums of reflexive modules are reflexive over such an algebra R , we may assume that R is a local uniserial algebra, so that the ideals of R are linearly ordered and every R -module is a direct sum of factors of R (see [1, Section 32]). In this case, $R/\ell_R(M)$ embeds in M and every subquotient of ${}_R M$ is a factor of $R/\ell_R(M)$. In the presence of these facts Theorem 1.2 yields the following corollary almost at once. We note, however, that [10, Theorem 1] is more general.

Corollary 1.5. *The direct sum of a finite number of reflexive modules over a split uniserial algebra is reflexive.*

Proof. Since we may assume that the ideals of R are linearly ordered, given modules M and N , we may also assume that $\ell_R(M) \subseteq \ell_R(N) \subseteq \ell_R(X)$ for every common subquotient X of M and N . So since $R/\ell_R(N)$ embeds in N , we see that both sides of the desired equality are equal to $\ell_R(N)$. □

Next, as further applications of this theorem, we shall see that it yields particularly nice proofs of some key results in [10].

In [10], in order to show that direct sums of reflexive modules may fail to be reflexive over a split K -algebra R with radical J whose quiver contains a triple arrow, a pair of reflexive modules was constructed with diagrams (as in [8])

$$M : \begin{array}{ccccc} & u & & v & \\ & a \downarrow & & b \searrow & \\ & x & & y & \end{array} \quad \text{and} \quad N : \begin{array}{ccc} & w & \\ & a, c \downarrow & \\ & z & \end{array}$$

with a, b, c linearly independent elements of $J \setminus J^2$. (The diagram indicates that $au = x, aw = cw = z$, etc.) Here

$$\ell_R(M) + \ell_R(N) = \ell_R(M) + K(a - c) + Kb,$$

but the only common subquotients of M and N are simple modules, so

$$\ell_R\left(\bigoplus_{i \in I} X_i\right) = \ell_R(M) + J \neq \ell_R(M) + \ell_R(N).$$

Thus the theorem shows that $M \oplus N$ is not reflexive.

In the positive vein we shall employ the following lemma and Theorem 1.2 to obtain a simple proof of a key part of [10, Proposition 4].

Lemma 1.6. *Let R be a split local K -algebra of $\dim(KR) \leq 3$. If M is a faithful left R -module, then every proper cyclic module is isomorphic to a subquotient of M .*

Proof. (This proof is essentially contained in the proof of [10, Proposition 4].) If R is uniserial, then ${}_R R$ embeds in M , so we may assume that $J^2 = 0$ and that $\dim(KJ) = 2$. Let $u \in M \setminus \text{Soc } M$ and suppose that $\ell_R(u) \neq 0$. Then, since $u \notin \text{Soc } M$, there is a $b \in J$ with $\ell_R(u) = Kb$. But then there is a $v \in M$ with $bv \neq 0$, and there is an $a \in J$ with $Ka = \ell_R(v)$. Thus it follows that

$$J = Ka \oplus Kb.$$

Now

$$Ru \cong R/Rb \quad \text{and} \quad Rv \cong R/Ra$$

are non-isomorphic uniserial submodules of M of composition length 2. If $Ru \cap Rv = 0$, we see that $\ell_R(u + v) = 0$. Otherwise

$$Ru \cap Rv = Kau = Kbv = \text{Soc}(Ru + Rv),$$

so the annihilator of any non-zero K -linear combination of u and v is properly contained in J , and we may assume that $au = bv$. Then since

$$(ka + lb)(k^{-1}u - \ell^{-1}v) = au - bv = 0,$$

we see that every proper cyclic R -module is a subquotient of M . □

Now we have the promised result of Fuller, Nicholson and Watters [10].

Proposition 1.7. *Let R be a split local K -algebra of $\dim(KR) \leq 3$. If M and N are reflexive R -modules, then so is $M \oplus N$.*

Proof. We may assume that $M \oplus N$ is faithful. If neither M nor N is faithful, then $J = Ka \oplus Kb$ with $aM = 0$ and $bN = 0$. But then

$$\ell_R(M) + \ell_R(N) = J = \ell_R(R/J)$$

and the desired equality holds. If M is faithful and N is not, then by Lemma 1.6, every cyclic subquotient of N is a subquotient of M , so

$$\ell_R(M) + \ell_R(N) = \ell_R(N) = \ell_R\left(\bigoplus_{i \in I} X_i\right).$$

Suppose both M and N are faithful. If R embeds in M , then every cyclic subquotient of N is a common subquotient of M and N , so

$$\ell_R(M) + \ell_R(N) \subseteq \ell_R\left(\bigoplus_{i \in I} X_i\right) = \ell_R(N) = 0.$$

Otherwise, the cyclic subquotients of M and N are all proper, so by Lemma 1.6, they are all common to both modules and again

$$\ell_R\left(\bigoplus_{i \in I} X_i\right) = 0.$$

□

2. CONTROLLING

The following the definition in [9] or [10], given bimodules ${}_R M_\Delta$ and ${}_R N_\Delta$, we say that M *controls* N in case for each pair (α, n) , with $\alpha \in \text{End}(N_\Delta)$ and $n \in N$, there is a set

$$C_{(\alpha, n)} = \{(m_i, n_i) \mid i \in I\} \subseteq M \times N,$$

called a *connection* for α and n , such that, if there are $r_i \in R$ with

$$r_i m_i = 0 \text{ and } \alpha n_i = r_i n_i$$

for all $i \in I$, then $\alpha n = 0$.

Controlling is a particularly useful concept. In [9] it was shown that if N is either generated or cogenerated by subquotients of M , then M controls N ; and that if M is reflexive and controls N , then $M \oplus N$ is reflexive. Here we shall present an equivalent version of controlling that yields a partial converse to this last assertion.

Proposition 2.1. *The bimodule ${}_R M_\Delta$ controls ${}_R N_\Delta$ if and only if, for each $0 \neq \alpha \in \text{End}(N_\Delta)$, there is a pair*

$$P_\alpha = (m_\alpha, n_\alpha) \in M \times N$$

with $\alpha n_\alpha \notin \ell_R(m_\alpha)n_\alpha$ (i.e., for all $r \in R$, if $\alpha n_\alpha = r n_\alpha$, then $r m_\alpha \neq 0$).

Proof. (\Rightarrow) Let $\alpha \in \text{End}(N_\Delta)$ with $\alpha n \neq 0$ and suppose that $C_{(\alpha, n)} = \{(m_i, n_i) \mid i \in I\}$ is a connection for α and n . If $\alpha n_i \in \ell_R(m_i)n_i$ for all $i \in I$, then there are r_i with $r_i m_i = 0$ and $\alpha n_i = r_i n_i$ for all $i \in I$. But $\alpha n \neq 0$, so there is a pair (m_i, n_i) with $\alpha n_i \notin \ell_R(m_i)n_i$.

(\Leftarrow) If $\alpha n = 0$, then any $C_{(\alpha,n)} = \{(m_i, n_i) \mid i \in I\}$ is a connection for α and n . If $\alpha n \neq 0$, let

$$C_{(\alpha,n)} = \{P_\alpha\} = \{(m_\alpha, n_\alpha)\}.$$

Then it never occurs that $r_\alpha m_\alpha = 0$ and $\alpha n_\alpha = r_\alpha n_\alpha$, so $C_{(\alpha,n)}$ is a connection. \square

It is worthy of note that when employing Proposition 2.1 to test for controlling, one only needs to consider those $0 \neq \alpha \in \text{alglat}(N)$. Indeed, if $0 \neq \alpha \in \text{End}(N_\Delta) \setminus \text{alglat}(N)$, then there is a n_α such that $\alpha n_\alpha \notin Rn_\alpha$, so for any m_α , $\alpha n_\alpha \notin \ell_R(m_\alpha)n_\alpha$. Also one easily checks that $\ell_R(M) \subseteq \ell_R(N)$ whenever M controls N .

The necessity part of the following corollary was established in [9], and has been employed in several subsequent papers. The second statement generalizes the fact that if M is reflexive and controls N , then $M \oplus N$ is reflexive.

Corollary 2.2. *The bimodule ${}_R M_\Delta$ controls ${}_R N_\Delta$ if and only if the restriction mapping*

$$\text{res} : \text{alglat}(M \oplus N) \rightarrow \text{alglat}(M)$$

is injective.

Moreover, if these conditions hold and $\text{Im}(\text{res}) \subseteq \lambda(R)$, then $M \oplus N$ is reflexive.

Proof. (\Leftarrow) Suppose M does not control N . Then there is an $\alpha \neq 0$ in $\text{End}(N_\Delta)$ such that for all $m \in M$ and all $n \in N$, there is an $r \in \ell_R(m)$ with $\alpha n = rn$. But then $0 \neq \beta = (0, \alpha) \in \text{End}((M \oplus N)_\Delta)$ and $\beta(m, n) = (0, \alpha n) = r(m, n)$, so $\beta \in \text{alglat}(M \oplus N)$, and $\beta|_M = 0$.

(\Rightarrow) If $\beta = (0, \alpha) \in \text{alglat}(M \oplus N)$ with $\alpha \neq 0$, then for each pair $(m, n) \in M \times N$ there is an $r \in R$ with $\beta(m, n) = (rm, rn) = (0, rn)$, contrary to the condition of Proposition 2.1.

For the last statement, assume that res is injective, $\delta \in \text{alglat}(M \oplus N)$ and $\text{res}(\delta) = \lambda_2(r)$ in the commutative diagram:

$$\begin{array}{ccc} \text{alglat}(M \oplus N) & \xrightarrow{\text{res}} & \text{alglat}(M) \\ \lambda_1 \swarrow & & \lambda_2 \nearrow \\ & R & \end{array}$$

Then $\text{res}(\delta) = \lambda_2(r) = \text{res}(\lambda_1(r))$, so $\delta = \lambda_1(r)$. \square

Suppose that M controls N . Then N is an $R/\ell_R(M)$ -module, and if M is reflexive, then so is $M \oplus N$. Conversely we have

Corollary 2.3. *If ${}_R M$ is faithful and $M \oplus N$ is reflexive, then M controls N . Thus if M is a faithful reflexive R -module, then $M \oplus N$ is reflexive if and only if M controls N .*

Proof. As in the previous proof, we have $\text{res} \circ \lambda_1 = \lambda_2$. Thus if λ_1 is epic and λ_2 is monic, then res is monic, so Corollary 2.2 applies. \square

A QF-3 algebra R is a finite dimensional algebra with a (*unique minimal faithful*) module U that embeds as a direct summand in every faithful module. In [9] it was shown that over a QF-2 algebra (a special type of QF-3 algebra, see [1, Section 31]) every faithful module is reflexive if U is reflexive; and the problem was posed of determining whether this is the case for QF-3 algebras. Some recent progress has been made by Snashall in [15]. Perhaps this last corollary may help to shed more light on this problem.

3. UNIVERSAL k -REFLEXIVITY

In [2] Azoff showed, from an operator theory point of view, that for an integer $k \geq 3$, if $\dim({}_\mathbb{C}M) \leq k$, then the direct sum $M^{(k-1)}$ of $k-1$ copies of M is reflexive. This topic was treated later using algebraic methods in [11]. We conclude by showing that for any left artinian ring R there is an integer k such that the direct sum of k copies of every bimodule ${}_R M_\Delta$ is reflexive. To do so we shall employ

Lemma 3.1. *If $R/\ell_R(M)$ embeds in N and ${}_R M_\Delta$ controls ${}_R N_\Delta$, then $M \oplus N$ is reflexive.*

Proof. Since $R/\ell_R(M)$ embeds in N , there is an $n_0 \in N$ with $\ell_R(n_0) = \ell_R(M)$. Let $\delta = (\beta, \gamma) \in \text{alglat}(M \oplus N)$ and suppose that $\gamma(n_0) = sn_0$. Then for any $m \in M$, there are $r, t \in R$ with

$$r(m, n_0) = \delta(m, n_0) = (\beta m, \gamma n_0) = (tm, sn_0),$$

so $r - s \in \ell_R(n_0) = \ell_R(M)$ and

$$\beta m = rm = sm.$$

Thus $\text{res}(\delta) = \beta = \lambda(s)$, and Corollary 2.2 applies. \square

Theorem 3.2. *For each left artinian ring R there is a positive integer k such that every $R - \Delta$ -bimodule is k -reflexive. Indeed if*

$$k = \sup\{c(\text{Soc}({}_R R/I)) \mid I \leq {}_R R\} + 1,$$

then $M^{(k)}$ is reflexive for every bimodule ${}_R M_\Delta$.

Proof. Let $c = c(\text{Soc}({}_R R/\ell_R(M)))$ and let $N = M^{(c)}$. Then it is easy to see that $R/\ell_R(M)$ embeds in N and, of course M controls N . Thus $M \oplus N = M^{(c+1)}$ is reflexive by Lemma 3.1. \square

This value of k cannot be improved since, over the uniserial K -algebra $R = K[x]/x^2$, the regular module ${}_R R$ is not reflexive (see [2] or [9]).

According to [11, Corollary 3] the result of Azoff mentioned above can be extended to the K -algebra-bimodule case. Thus from this result and the proof of Theorem 3.2 we have

Corollary 3.3. *Let R and Δ be finite dimensional K -algebras and let M be an $R - \Delta$ -bimodule such that ${}_R M$ is faithful. If $k \geq 2$ and*

$$\min\{c(\text{Soc}({}_R R)) + 1, c(M_\Delta) - 1\} \leq k,$$

then $M^{(k)}$ is reflexive.

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