

INCOMPRESSIBLE SURFACES IN HANDLEBODIES AND CLOSED 3-MANIFOLDS OF HEEGAARD GENUS 2

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ABSTRACT. In this paper, we shall prove that for any integer $n > 0$, 1) a handlebody of genus 2 contains a separating incompressible surface of genus n , 2) there exists a closed 3-manifold of Heegaard genus 2 which contains a separating incompressible surface of genus n .

1. INTRODUCTION

Let M be a 3-manifold, and let F be a properly embedded surface in M . F is said to be compressible if either F is a 2-sphere and F bounds a 3-cell in M , or there exists a disk $D \subset M$ such that $D \cap F = \partial D$, and ∂D is nontrivial on F . Otherwise F is said to be incompressible.

W. Jaco (see [3]) has proved that a handlebody of genus 2 contains a nonseparating incompressible surface of arbitrarily high genus, and asked the following question.

Question A. Does a handlebody of genus 2 contain a separating incompressible surface of arbitrarily high genus?

In the second section, we shall give an affirmative answer to this question. The main result is the following.

Theorem 2.7. *A handlebody of genus 2 contains a separating incompressible surface S of arbitrarily high genus such that $|\partial S| = 1$.*

If M is a closed 3-manifold of Heegaard genus 1, then M is homeomorphic to either a lens space or $S^2 \times S^1$. In the third section, we shall prove that the Heegaard genus of a closed 3-manifold M does not limit the genus of an incompressible surface in M . The main result is the following.

Theorem 3.10. *For any integer $n > 0$, there exists a closed 3-manifold of Heegaard genus 2 which contains a separating incompressible surface of genus n .*

If X is a manifold, we shall denote by ∂X the boundary of X , and by $|\partial X|$ the number of components of ∂X . If $F(x_1, \dots, x_n)$ is a free group, and y is an element

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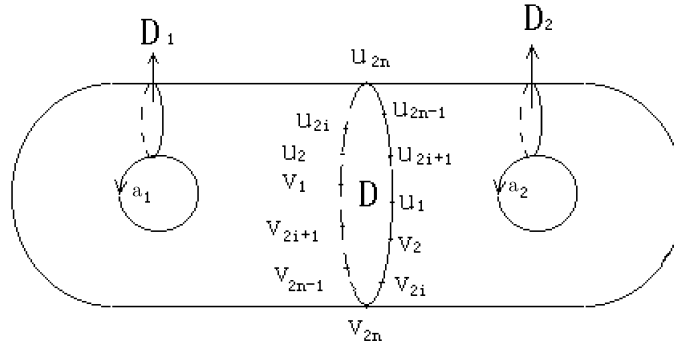


FIGURE 1.

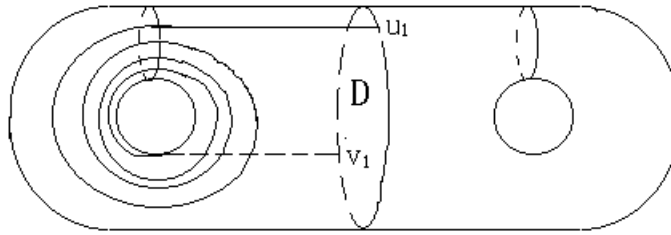


FIGURE 2.

in F , we shall denote by $L(y)$ the minimal length of y with respect to the basis x_1, \dots, x_n .

Some examples answering Jaco's question have also been given by Hugh Howards in "Generating disjoint incompressible surfaces", preprint, 1998.

2. INCOMPRESSIBLE SURFACES IN HANDLEBODIES

Let H_2 be a handlebody of genus 2, and let (D_1, D_2) be a set of basis disks of H_2 . Let D be a separating disk of H_2 such that D_1 and D_2 lie on opposite sides of D . Suppose that $u_1, \dots, u_{2n}, v_1, \dots, v_{2n}$ are $4n$ points on ∂D as in Figure 1.

Suppose that $u_1v_1, \dots, u_{2n}v_{2n}$ are $2n$ arcs on ∂H_2 such that

- 1) u_1v_1 is as in Figure 2,
- 2) $u_{2i}v_{2i-1}$ and $u_{2i-1}v_{2i}$ are as in Figure 3,
- 3) $u_{2i+1}v_{2i}$ and $u_{2i}v_{2i+1}$ are as in Figure 3, and
- 4) u_kv_k is the union of u_kv_{k-1} , $v_{k-1}u_{k-1}$ and $u_{k-1}v_k$.

Then $u_iv_i \subset u_{i+1}v_{i+1}$.

Suppose that $N_k = u_kv_k \times B_k$, where B_k is a half disk in H_2 (as in Figure 4) such that

- 1) $\{u_i\} \times B_k \cup \{v_i\} \times B_k \subset D (i \leq k)$, and
- 2) if $x \in u_kv_k$, then $\{x\} \times B_i \subset \text{int}(\{x\} \times B_k) (i > k)$.

Let $C_0 = D - \bigcup_{i=1}^k (\{u_i\} \times B_i) - \bigcup_{i=1}^k (\{v_i\} \times B_i)$, and $D_0 = \bar{C}_0$. Let $C_i = \partial(u_iv_i \times B_i) - \partial H_2 - \{u_i\} \times B_i - \{v_i\} \times B_i$, and $D_i = \bar{C}_i$, where $1 \leq i \leq k$. Let $S_k = \bigcup_{i=0}^k D_i$. Then S_k is a properly embedded surface in H_2 ($1 \leq k \leq 2n$). Let

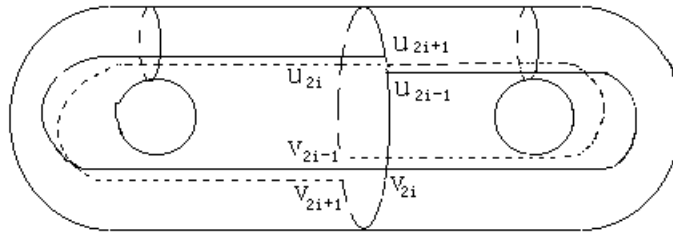


FIGURE 3.

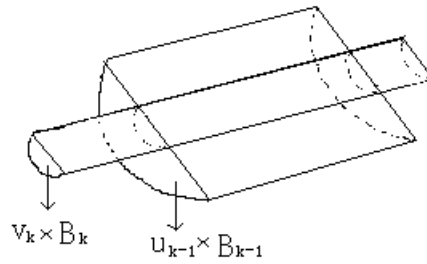


FIGURE 4.

$A_i = D_0 \cup D_i$ ($i > 0$). Then A_i is an annulus. Hence S_k is a union of k annuli A_1, \dots, A_k along D_0 .

Lemma 2.1. $|\partial S_{2k-1}| = 2$ and $|\partial S_{2k}| = 1$ ($1 \leq k \leq n$).

Proof. It is clear that $|\partial S_1| = 2$. Since u_2 and v_2 lie in the two distinct components of ∂S_1 , by construction, $|\partial S_2| = 1$. We can prove that $|\partial S_{2k-1}| = 2$ and $|\partial S_{2k}| = 1$ by induction, for $1 \leq k \leq n$. \square

Lemma 2.2. *The genus of S_{2n} is n .*

Proof. By construction, S_{2n} is a union of $2n$ annuli A_1, \dots, A_{2n} along D_0 . Hence $\pi_1(S_{2n}) = F(x_1, \dots, x_{2n})$, where x_i is represented by the core of A_i . Since $|\partial S_{2n}| = 1$, the genus of S_{2n} is n . \square

Let $i : S_{2n} \rightarrow H_2$ be the inclusion map, and $i_* : \pi_1(S_{2n}) \rightarrow \pi_1(H_2)$ be the map induced by i . Suppose that a_1 and a_2 are the two generators of $\pi_1(H_2)$ shown in Figure 1. Then by construction we have

$$\begin{aligned} i_*(x_1) &= a_1^4, \\ i_*(x_2) &= a_2^{-1} a_1^{-4} a_2^{-1}, \\ &\vdots \\ &\vdots \\ i_*(x_{2i}) &= a_2^{-1} (a_2 a_1)^{1-i} a_1^{-4} (a_1 a_2)^{1-i} a_2^{-1}, \\ i_*(x_{2i+1}) &= (a_1 a_2)^i a_1^4 (a_2 a_1)^i, \\ &\vdots \\ &\vdots \\ i_*(x_{2n}) &= a_2^{-1} (a_2 a_1)^{1-n} a_1^{-4} (a_1 a_2)^{1-n} a_2^{-1}. \end{aligned}$$

Let y be a nontrivial element of $\pi_1(S_{2n})$. Then $y = \prod_{i=1}^m b_i$, where $b_i = \prod_{j=1}^{m_i} b_{ij}^{p_{ij}}$, $b_{ij} \in \{x_1, x_2, \dots, x_{2n}\}$ and $m_i \geq 1$, such that

- 1) $p_{ij} \neq 0$,
- 2) for each i , $1 \leq i \leq m$, either all the b_{ij} 's ($1 \leq j \leq m_i$) belong to $\{x_1, x_3, \dots, x_{2n-1}\}$ or they all belong to $\{x_2, x_4, \dots, x_{2n}\}$,
- 3) the b_{ij} 's belong to $\{x_1, x_3, \dots, x_{2n-1}\}$ if and only if the b_{i+1j} 's belong to $\{x_2, x_4, \dots, x_{2n}\}$, and
- 4) $b_{ij} \neq b_{ij+1}^{\pm 1}$.

Lemma 2.3. *If $b_{ij} \in \{x_1, \dots, x_{2n-1}\}$ for $j \in \{1, \dots, m_i\}$, then $L(i_*(b_i)) > 1$. Also, the first letter of $i_*(b_i)$ is a_1 or a_1^{-1} , and the last letter of $i_*(b_i)$ is a_1 or a_1^{-1} .*

Proof. Suppose that $b_{ij} = x_{2l_j+1}$, where $l_j \in \{0, 1, \dots, n-1\}$. Then

$$i_*(b_{ij}^{p_{ij}}) = (a_1 a_2)^{l_j} a_1^4 (a_2 a_1)^{l_j} \dots (a_1 a_2)^{l_j} a_1^4 (a_2 a_1)^{l_j}, \text{ or}$$

$$i_*(b_{ij}^{p_{ij}}) = (a_2 a_1)^{-l_j} a_1^{-4} (a_1 a_2)^{-l_j} \dots (a_2 a_1)^{-l_j} a_1^{-4} (a_1 a_2)^{-l_j}.$$

If $p_{ij} > 0$, and $p_{ij+1} > 0$, then

$$i_*(b_{ij} b_{ij+1}) = (a_1 a_2)^{l_j} a_1^4 (a_2 a_1)^{l_j} (a_1 a_2)^{l_{j+1}} a_1^4 (a_2 a_1)^{l_{j+1}}.$$

If $p_{ij} > 0$, and $p_{ij+1} < 0$, then

$$i_*(b_{ij} b_{ij+1}) = (a_1 a_2)^{l_j} a_1^4 (a_2 a_1)^{l_j} a_1^{-4} (a_1 a_2)^{-l_{j+1}}.$$

Since $b_{ij} \neq b_{ij+1}$, $l_j \neq l_{j+1}$. It is easy to see that the first and last letters of $i_*(b_i)$ are a_1 or a_1^{-1} , and $L(i_*(b_i)) \geq 2$. □

Lemma 2.4. *If $b_{ij} \in \{x_2, \dots, x_{2n}\}$, then $L(i_*(b_i)) > 1$. Also, the first letter of $i_*(b_i)$ is a_2 or a_2^{-1} , and the last letter of $i_*(b_i)$ is a_2 or a_2^{-1} .*

Proof. The proof of Lemma 2.4 is similar to the proof of Lemma 2.3. □

Lemma 2.5. S_{2n} is incompressible in H_2 .

Proof. Suppose that y is a nontrivial element of $\pi_1(S_{2n})$. Then $y = \prod_{i=1}^m b_i$, where b_i satisfies the above conditions. By Lemma 2.3 and Lemma 2.4, $L(i_*(y)) = \sum_{i=1}^m L(i_*(b_i))$. Hence $L(i_*(y)) > 1$, and S_{2n} is incompressible in H_2 . □

Lemma 2.6. S_{2n} is separating in H_2 .

Proof. By construction, $[\partial S_{2n}] = 0$ in $H_1(\partial H_2)$. Hence ∂S_{2n} is separating on ∂H_2 , and S_{2n} is separating in H_2 . □

Theorem 2.7. *A handlebody of genus 2 contains a separating incompressible surface S of arbitrarily high genus such that $|\partial S| = 1$.*

Proof. Let n be a positive integer, then S_{2n} is a separating incompressible surface in H_2 by the above argument. □

Remark 1. In fact a handlebody of genus 2 also contains a separating incompressible surface S of arbitrarily high genus such that $|\partial S| = 2$. For example, S_{2n-1} is a separating incompressible surface of genus $n-1$ in H_2 .

Remark 2. For any positive integer n , there exist infinitely many separating incompressible surface of genus n in a handlebody of genus 2. For example, let $u_1 v_1$ intersect D_1 in m points where $m \geq 3$. Then by the proof of Theorem 2.7, we can obtain another separating incompressible surface of genus n in H_2 by the same method as in the above construction.

Corollary 2.8. *For any integer $n > 1$, a handlebody of genus n contains a separating incompressible surface S of arbitrarily high genus.*

3. INCOMPRESSIBLE SURFACES IN CLOSED 3-MANIFOLDS
OF HEEGAARD GENUS 2

Let M be a compact 3-manifold with boundary. If c_1, \dots, c_n are disjoint simple closed curves on ∂M , we denote by $\tau(M, \bigcup_{i=1}^n c_i)$ the manifold obtained by attaching 2-handles to M along disjoint regular neighborhoods of c_1, \dots, c_n , and $M[c_1] \dots [c_n]$ the manifold obtained by capping off possible 2-sphere components of $\partial\tau(M, \bigcup_{i=1}^n c_i)$. If c is a nontrivial simple closed curve on a toral component of ∂M , we denote by $M(c)$ the manifold $M[c]$. Now if $F (\neq S^2)$ is a separating incompressible closed surface in $\tau(M, \bigcup_{i=1}^n c_i)$, then F is also a separating incompressible surface in $M[c_1] \dots [c_n]$.

If S is a properly embedded surface in M , we denote by \hat{S} the surface obtained by capping off the boundary components of S with disks in $\tau(M, \partial S)$.

Lemma 3.1. *H_2 contains no closed incompressible surface.*

Proof. Let D be a properly embedded disk in H_2 such that ∂D is nontrivial on ∂H_2 . If F is a closed incompressible surface, then F may be isotoped to be disjoint from D . Hence a 3-cell contains a closed incompressible surface, a contradiction. \square

Lemma 3.2. *Let M be a 3-manifold, and let J be a simple closed curve on ∂M such that $\partial M - J$ is incompressible. If M has compressible boundary, then $\tau(M, J)$ is a ∂ -irreducible manifold.*

Proof. See [4, Theorem 2]. \square

Let H_2 be a handlebody of genus 2, and let S be a separating incompressible surface of genus $n > 0$ in H_2 such that $|\partial S| = 1$.

Lemma 3.3. *$\tau(H_2, \partial S)$ is a ∂ -irreducible 3-manifold, and \hat{S} is a separating closed incompressible surface of genus n in $\tau(H_2, \partial S)$.*

Proof. Suppose that S separates H_2 into H, H' and ∂S separates ∂H_2 into T, T' , such that $\partial H = S \cup T$ and $\partial H' = S \cup T'$.

Claim 1. $\partial H(\partial H')$ is compressible in $H(H')$.

Proof. If ∂H is incompressible in H , then ∂H is incompressible in H_2 , contradicting Lemma 3.1. \square

Claim 2. $T(T')$ is incompressible in $H(H')$.

Proof. If T is compressible in H , then there exists a nontrivial simple closed curve c on T such that c bounds a disk D in H . Since the genus of ∂H_2 is 2, T is a once punctured torus whose boundary is isotopic to ∂S . Hence ∂S bounds a disk in H_2 , a contradiction. \square

Since $\partial H(\partial H')$ is compressible in $H(H')$, and S and $T(T')$ are incompressible in $H(H')$, it follows that $\tau(H, \partial S)$ and $\tau(H', \partial S)$ are ∂ -irreducible 3-manifolds by Lemma 3.2.

Since $\tau(H_2, \partial S) = \tau(H, \partial S) \cup \tau(H', \partial S)$, \hat{S} is a separating incompressible closed surface in $\tau(H_2, \partial S)$. \square

Since the genus of ∂H_2 is 2, $\partial\tau(H_2, \partial S)$ consists of two tori, T_1 and T_2 , say.

Let M be a 3-manifold with one of component T of ∂M a torus. If r_1 and r_2 are two slopes on T , we shall denote by $\Delta(r_1, r_2)$ the minimal geometric intersection number among all the curves representing the slopes.

Lemma 3.4. *Let M be a ∂ -irreducible 3-manifold with one component T of ∂M a torus, and let F be a closed incompressible surface in M which is not parallel to T . If r_1 and r_2 are two slopes on T such that F is compressible in $M(r_1)$ and $M(r_2)$, then either*

- 1) $\Delta(r_1, r_2) \leq 1$, or
- 2) there exists a slope r on T such that $\Delta(r, r_1) \leq 1$ and $\Delta(r, r_2) \leq 1$.

Proof. See [6, Theorem 1].

Corollary 3.5. *Let M be a ∂ -irreducible 3-manifold with one component T of ∂M a torus. If F is a closed incompressible surface in M which is not parallel to T , then there exists a nontrivial simple closed curve c on T such that F is incompressible in $M(c)$.*

Lemma 3.6. *There exist two nonseparating simple closed curves c_1 and c_2 on ∂H_2 such that c_i is disjoint from ∂S , and \hat{S} is a separating incompressible surface in $H_2[\partial S][c_1][c_2]$.*

Proof. Suppose that \hat{S} separates $\tau(H_2, \partial S)$ into M_1 and M_2 such that $T_1 \subset M_1$ and $T_2 \subset M_2$. By the proof of Lemma 3.1, M_i is ∂ -irreducible. Since \hat{S} is not parallel to T_i in M_i , by Corollary 3.5, there exists a simple closed curve c_i ($1 \leq i \leq 2$) on T_i such that \hat{S} is incompressible in $M_i(c_i)$.

By an isotopy, we can suppose that c_i is disjoint from ∂S . Hence \hat{S} is incompressible in $H_2[\partial S][c_1][c_2]$. It is easy to see that \hat{S} is separating in $H_2[\partial S][c_1][c_2]$. \square

Definition 3.7. Two simple closed curves α and β on ∂M are said to be coplanar if some component of $\partial M - \alpha \cup \beta$ is an annulus or a once punctured annulus.

Lemma 3.8. *Suppose that α is a nonseparating curve on ∂M . If a separating curve β on ∂M is coplanar to α , then $M[\alpha] = M[\beta][\alpha]$.*

Proof. See [5, Lemma 5.1]. \square

Lemma 3.9. $H_2[\partial S][c_1][c_2] = H_2[c_1][c_2]$.

Proof. Since c_1 is coplanar to ∂S on ∂H_2 , we have $H_2[\partial S][c_1][c_2] = H_2[c_1][c_2]$. \square

Theorem 3.10. *For any integer $n > 0$, there exists a closed 3-manifold M of Heegaard genus 2 which contains a closed separating incompressible surface of genus n .*

Proof. Let H_2 be a handlebody of genus 2, and let S be a separating incompressible surface of genus n such that $|\partial S| = 1$. Then $H_2[c_1][c_2]$ contains a separating incompressible surface \hat{S} of genus n , where c_1 and c_2 are disjoint nonseparating simple closed curves on ∂H_2 as in Lemma 3.6. Obviously the Heegaard genus of $H_2[c_1][c_2]$ is 2. \square

Corollary 3.11. *Suppose that $m \geq 2$. Then for any integer $n > 0$ there exists a closed 3-manifold of Heegaard genus m which contains a closed separating incompressible surface of genus n .*

In fact there are infinitely many simple closed curves c on ∂H_2 such that $\tau(H_2, c)$ contains a closed separating incompressible surface of genus n . This is shown by the following example.

Example. Let H_2 be a handlebody of genus 2. Let S_{2m} be an incompressible surface of genus n constructed by the same method as the construction of S_{2n} (as in Section 2), such that u_1v_1 intersects D_1 in m points. Then \hat{S}_{2m} is incompressible in $\tau(H_2, \partial S_{2m})$.

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