PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 10, Pages 2907–2909
S 0002-9939(00)05362-4
Article electronically published on March 2, 2000

APPROXIMATION OF FIXED POINTS OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS ON ARBITRARY CLOSED, CONVEX SETS IN A BANACH SPACE

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(Communicated by Dale Alspach)

ABSTRACT. We show that any fixed point of a Lipschitzian, strictly pseudocontractive mapping T on a closed, convex subset K of a Banach space X is necessarily unique, and may be norm approximated by an iterative procedure. Our argument provides a convergence rate estimate and removes the boundedness assumption on K, generalizing theorems of Liu.

Let $(X, \|\cdot\|)$ be a Banach space. Let K be a non-empty closed, convex subset of X and $T\colon K \to K$. We will assume that T is *Lipschitzian*, i.e. there exists $L > 0$ such that

$$\|T(x) - T(y)\| \leq L\|x - y\|,$$

for all $x, y \in K$. Of course, we are most interested in the case where $L \geq 1$.

We also assume that T is *strictly pseudocontractive*. Following Liu [1] this may be stated as: there exists $k \in (0, 1)$ for which

$$\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\|,$$

for all $r > 0$ and all $x, y \in K$.

Throughout, $\mathbf{N}$ will denote the set of positive integers.

The following results generalize Liu [1, Theorems 1 and 2], because we remove the assumption that K is bounded and we provide a general convergence rate estimate. We note in passing, however, that the proof of Theorem 2 of Liu [1] does not use the stated boundedness assumption. Our results still extend this enhanced version of Liu [1, Theorem 2], by improving the convergence rate estimate.

Theorem 1. *Let $(X, \|\cdot\|), K, T, L$ and k be as described above. Let $q \in K$ be a fixed point of T. Suppose that $(\alpha_n)_{n\in\mathbf{N}}$ is a sequence in $(0, 1]$ such that for some $\eta \in (0, k)$, for all $n \in \mathbf{N}$,*

$$\alpha_n \leq \frac{k - \eta}{(L + 1)(L + 2 - k)}; \quad \textit{while } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Received by the editors May 4, 1998 and, in revised form, November 2, 1998.
1991 *Mathematics Subject Classification.* Primary 47H17.
Key words and phrases. Banach space, Lipschitzian mapping, strictly pseudocontractive mapping, fixing points.
This research was supported by UGC, India, Grant No. U4/4997/97-98.

Fix $x_1 \in K$. Define, for all $n \in \mathbf{N}$,

$$x_{n+1} := (1-\alpha_n)x_n + \alpha_n T(x_n).$$

Then there exists $(\beta_n)_{n\in\mathbf{N}}$, a sequence in $(0,1)$ with each $\beta_n \geq (\eta/(1+k))\alpha_n$, such that for all $n \in \mathbf{N}$,

$$\|x_{n+1} - q\| \leq \prod_{j=1}^{n}(1-\beta_j)\|x_1 - q\|.$$

In particular, $(x_n)_{n\in\mathbf{N}}$ converges strongly to q, and q is the unique fixed point of T.

Proof. Define $\delta_n := \|x_n - q\|$ for each $n \in \mathbf{N}$. Consider any $n \in \mathbf{N}$. Just as in the proof of Liu [1, Theorem 1], it follows that

$$\delta_n \geq (1+\alpha_n)\delta_{n+1} - (1-k)\alpha_n\delta_n - (2-k)\alpha_n^2\|x_n - T(x_n)\| - L(L+1)\alpha_n^2\delta_n. \tag{1}$$

Now, as noted in the proof of Liu [1, Theorem 2],

$$\|x_n - T(x_n)\| \leq (L+1)\delta_n. \tag{2}$$

Thus, from (1) and (2) we see that

$$\delta_{n+1} \leq \frac{A_n}{B_n}\delta_n, \tag{3}$$

where $A_n := 1 + (1-k)\alpha_n + (2-k+L)(L+1)\alpha_n^2$ and $B_n := 1+\alpha_n$. Define $\beta_n := 1 - A_n/B_n$. Then

$$\beta_n = \frac{\alpha_n}{1+a_n}[k - (L+1)(L+2-k)\alpha_n] \geq \frac{\alpha_n}{1+\alpha_n}\eta \geq \frac{\eta}{1+k}\alpha_n.$$

Further, from (3) we have

$$\delta_{n+1} \leq \frac{A_n}{B_n}\cdot\dots\cdot\frac{A_1}{B_1}\delta_1 = \prod_{j=1}^{n}(1-\beta_j)\delta_1.$$

Clearly, $\sum_{n=1}^{\infty}\beta_n = \infty$, and so $\prod_{j=1}^{\infty}(1-\beta_j) = 0$. Thus $x_n \to q$ in norm as $n \to \infty$, and q is the unique fixed point of T. □

Immediately we have two corollaries.

Corollary 1. *Let $(X, \|\cdot\|), K, T, L, k, q$ and $(x_n)_{n\in\mathbf{N}}$ be as in the hypotheses of Theorem 1, where $(\alpha_n)_{n\in\mathbf{N}}$ is a sequence in $(0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n = \infty$; and $\alpha_n \to 0$. Then $(x_n)_{n\in\mathbf{N}}$ converges strongly to q, and q is the unique fixed point of T.*

Corollary 2. *Let $(X, \|\cdot\|), K, T, L, k, q, \eta$ and $(x_n)_{n\in\mathbf{N}}$ be as in the hypotheses of Theorem 1, where $(\alpha_n)_{n\in\mathbf{N}}$ is the sequence in $(0,1]$ given for every $n \in \mathbf{N}$ by*

$$\alpha_n := \frac{k-\eta}{(L+1)(L+2-k)}.$$

Then we have the following geometric convergence rate estimate for $(x_n)_{n\in\mathbf{N}}$: for all $n \in \mathbf{N}$,

$$\|x_{n+1} - q\| \leq \rho^n\|x_1 - q\|,$$

where

$$\rho := 1 - \beta_1 = 1 - \eta\frac{\alpha_1}{1+\alpha_1}.$$

Finally, we remark that the choice $\eta := k/2$ yields

$$\rho = \rho_0 := 1 - \frac{k^2}{4(L+1)(L+2-k)+2k}.$$

The minimal ρ value of Corollary 2 as η varies over $(0,k)$ is less than or equal to ρ_0. Thus it is less than the ρ value of Liu [1, Theorem 2]:

$$\rho = 1 - \frac{k^2}{4(3+3L+L^2)}.$$

Acknowledgements

We would like to thank the referee for his valuable comments and suggestions which helped in shaping our paper to the present form.

Reference

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