

**A WEIGHTED UNIFORM L^p -ESTIMATE
OF BESSEL FUNCTIONS: A NOTE ON A PAPER OF GUO**

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ABSTRACT. An improved Guo's uniform L^p estimate of Bessel functions is shown by using a uniform pointwise bound of Barceló and Córdoba.

Recently, Guo has shown, [Guo, Theorem 3.5], the following uniform L^p estimate:

$$(1) \quad \int_0^\infty |J_\nu(x)|^p x dx \leq C(p-4)^{-1}, \quad \nu \geq 0, p > 4.$$

Here $J_\nu(x)$ denotes the Bessel function of the first kind of order ν , cf. [W]. This estimate was proved first for $\nu = 0, 1, \dots$, by means of a dual form of a Fourier restriction theorem for the plane unit circle and then extended to an arbitrary $\nu \geq 0$. The estimate was crucial in proving the main result of [Guo], Theorem 4.1.

It was quite reasonable to expect a proof of (1) based on intrinsic properties of Bessel functions. Furthermore, it was natural to expect an estimate like (1) for a larger range of p 's by adding an appropriate power weight in the integral on the left side of (1). More precisely, it was natural to look for an inequality of the form

$$(2) \quad \int_0^\infty |J_\nu(x)|^p x^a dx \leq C(p, a), \quad \nu \geq 0,$$

with a constant $C(p, a) > 0$ depending only on p and a (we did not care about making the constant $C(p, a)$ the best possible).

Since $J_\nu(x) = O(x^{-1/2})$, $x \rightarrow \infty$, the necessary assumption on a to make the integral in (2) convergent at infinity for every single $\nu \geq 0$ is $a < p/2 - 1$. On the other hand $J_\nu(x) = O(x^\nu)$, $x \rightarrow 0$; hence the necessary assumption on a to make the integral in (2) convergent at zero for every $\nu \geq 0$ is $a > -1$.

It is now interesting to note that Guo's result, (1), shows that the assumption $-1 < a < p/2 - 1$ is also sufficient for (2) to hold in the case $0 < p \leq 4$. Indeed, assume

$$\int_1^\infty |J_\nu(x)|^q x dx \leq C_q, \quad \nu \geq 0,$$

holds true for every $q > 4$ and consider p and a such that $0 < p \leq 4$ and $a < p/2 - 1$. Since $2(a + 1) < p \leq 4$, we can choose $s > 1$ satisfying $2(a + 1)s < 4 < ps$. Then,

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because $2(a + 1)s < 4$ implies $(a - 1)s' + 1 < -1$, by Hölder's inequality we obtain

$$\int_1^\infty |J_\nu(x)|^p x^a dx \leq \left(\int_1^\infty |J_\nu(x)|^{ps} x dx \right)^{1/s} \left(\int_1^\infty x^{(a-1)s'} x dx \right)^{1/s'} \leq A(C_{ps})^{1/s}.$$

In a similar way, assuming

$$\int_0^1 |J_\nu(x)|^q x dx \leq D_q, \quad \nu \geq 0,$$

is satisfied for every $q > 4$ and taking p and a such that $0 < p \leq 4$ and $a > -1$, we obtain

$$\int_0^1 |J_\nu(x)|^p x^a dx \leq B(D_{ps})^{1/s},$$

where, this time $s > 1$ is chosen in such a way that $ps > 4$ and $(a + 1)s > 2$. The main result of this note claims that, under suitable restrictions on a , (2) is valid for any p , $0 < p < \infty$.

Proposition. *Let $0 < p < \infty$ and $-1 < a < \frac{p}{2} - 1$ when $0 < p \leq 4$ or $-1 < a < \frac{p}{3} - \frac{1}{3}$ in the case $4 < p < \infty$. Then the uniform estimate*

$$(3) \quad \int_0^\infty |J_\nu(x)|^p x^a dx \leq C(p, a), \quad \nu \geq 0,$$

holds true.

The proof of the proposition is based on the following, uniform on $\nu \geq 2$, pointwise bounds for the Bessel functions (C and d are positive constants):

$$(4) \quad |J_\nu(x)| \leq C \begin{cases} \exp(-d\nu), & 0 < x < \nu/2, \\ \nu^{-1/4}(|x - \nu| + \nu^{1/3})^{-1/4}, & \nu/2 < x < 2\nu, \\ x^{-1/2}, & 2\nu < x < \infty. \end{cases}$$

The estimate (4) on the interval $0 < x < \nu/2$ is a consequence of

$$\Gamma(\nu + 1)(x/2)^{-\nu} |J_\nu(x)| \leq 1, \quad x > 0,$$

(and Stirling's formula) that holds for every $\nu \geq -1/2$ [W, p. 49 (1)], while on the two other intervals it is a consequence of bounds done by Barceló and Córdoba (see [BC, p. 66] or [C, p. 24]; cf. also [Va, p. 70]).

Proof of the Proposition. The left side of (3) is a continuous function of the variable $\nu \geq 0$; hence we can assume ν to be large, say $\nu \geq 2$. Given $\nu \geq 2$ we split the integration in (3) onto the intervals $(0, \nu/2)$, $(\nu/2, 2\nu)$ and $(2\nu, \infty)$. Then

$$\int_{2\nu}^\infty |J_\nu(x)|^p x^a dx \leq C \int_{2\nu}^\infty x^{a-p/2} dx = C_1 \nu^{a-p/2+1} \leq C_2$$

for $\nu \geq 2$ and p and a satisfying $a < p/2 - 1$. Also,

$$\int_0^{\nu/2} |J_\nu(x)|^p x^a dx \leq C \exp(-p d \nu) \int_0^{\nu/2} x^a dx \leq C_3$$

for $\nu \geq 2$ when a satisfies $a > -1$. On the interval $(\nu/2, 2\nu)$ we consider only the integration over $(\nu, 2\nu)$; the integration over $(\nu/2, \nu)$ can be treated analogously. We have

$$(5) \quad \int_{\nu}^{2\nu} |J_{\nu}(x)|^p x^a dx \leq C\nu^{a-p/4} \int_{\nu}^{2\nu} (x - \nu + \nu^{1/3})^{-p/4} dx.$$

If $0 < p \leq 4$, we evaluate the last integral and bound the right side of (5) by $C\nu^{a-p/2+1}$ when $p < 4$ or, by $C\nu^{a-1} \log \nu$ when $p = 4$. Both bounds are small for large ν by the assumption made on a . If $p > 4$, evaluating the last integral gives the bound $C\nu^{a-p/3+1/3}$ for the right side of (5) which is also correct by the assumption made on a . This finishes the proof of the proposition. \square

Remark. In fact, using the asymptotics of [BC, p. 66] leads to precise asymptotics of weighted L^p norms of the Bessel functions. Let $1 \leq p \leq \infty$, $\alpha < \frac{1}{2} - \frac{1}{p}$ and $\nu \rightarrow \infty$. Then (for $p = \infty$ one has to take $\sup_{x>0} |J_{\nu}(x)x^{\alpha}|$ as the L^{∞} norm)

$$(6) \quad \left(\int_0^{\infty} |J_{\nu}(x)x^{\alpha}|^p dx \right)^{1/p} \sim \begin{cases} \nu^{\alpha-1/2+1/p}, & 1 \leq p < 4, \\ \nu^{\alpha-1/4}(\log \nu)^{1/4}, & p = 4, \\ \nu^{\alpha-1/3+1/(3p)}, & 4 < p \leq \infty. \end{cases}$$

Here $f(\nu) \sim g(\nu)$ as $\nu \rightarrow \infty$ stands for $f(\nu) = O(g(\nu))$ and $g(\nu) = O(f(\nu))$ as $\nu \rightarrow \infty$. The upper bound in (6) is obtained, as in the proof of the proposition, by dividing $(0, \infty)$ into three different subintervals, majorizing the integrand and comparing the occurring bounds. The lower bound in (6) is a consequence of the aforementioned precise asymptotics of [BC].

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