

A REFINEMENT OF THE TORAL RANK CONJECTURE FOR 2-STEP NILPOTENT LIE ALGEBRAS

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ABSTRACT. It is known that the total (co)-homology of a 2-step nilpotent Lie algebra \mathfrak{g} is at least $2^{|\mathfrak{z}|}$, where \mathfrak{z} is the center of \mathfrak{g} . We improve this result by showing that a better lower bound is 2^t , where $t = |\mathfrak{z}| + \left\lceil \frac{|v|+1}{2} \right\rceil$ and v is a complement of \mathfrak{z} in \mathfrak{g} . Furthermore, we provide evidence that this is the best possible bound of the form 2^t .

1. INTRODUCTION

An outstanding conjecture, known as the *Toral Rank Conjecture* (TRC) claims that for any nilpotent Lie algebra \mathfrak{g} (over \mathbf{R} or \mathbf{C}) the total (co)-homology, with trivial coefficients, satisfies the inequality

$$|\mathrm{H}_*(\mathfrak{g})| \geq 2^{|\mathfrak{z}|},$$

where $\mathfrak{z} = \text{center}(\mathfrak{g})$.

The TRC is due to S. Halperin ([5], 1987). In 1988, Deninger and Singhof [4] proved it for 2-step nilpotent Lie algebras. Besides this class, a few special cases have been added recently. For example, it was shown in [2] that the TRC holds for \mathfrak{g} if its center has dimension ≤ 5 or has codimension ≤ 7 .

It turns out that, in general, $2^{|\mathfrak{z}|}$ is quite a lot smaller than $|\mathrm{H}_*(\mathfrak{g})|$, especially when $|\mathfrak{z}|$ is comparatively small compared to $|\mathfrak{g}|$.

In this short note we give a new lower bound for $|\mathrm{H}_*(\mathfrak{g})|$, when \mathfrak{g} is a 2-step nilpotent Lie algebra, that involves both the dimension of the center of \mathfrak{g} and its codimension. Furthermore, by using existing calculations, we show that actually it is the best possible general lower bound of the form 2^t .

Precisely, we give a direct proof of the following

Theorem. *Let \mathfrak{g} be a 2-step nilpotent Lie algebra of finite dimension over \mathbf{R} or \mathbf{C} . Let v be any direct complement, as vector spaces, of $\mathfrak{z} = \text{center}(\mathfrak{g})$. Then,¹*

$$|\mathrm{H}_*(\mathfrak{g})| \geq 2^{|\mathfrak{z}| + \left\lceil \frac{|v|+1}{2} \right\rceil}.$$

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¹The right brackets denote the integral part.

2. THE PROOF OF THE THEOREM

We will make use of the following combinatorial result.

Lemma. *Let n be a positive integer. Then,*

$$\left| \sum_{j=0}^n (-1)^j \binom{n}{2j} \right| + \left| \sum_{j=0}^n (-1)^j \binom{n}{2j+1} \right| = 2^{\lfloor \frac{n+1}{2} \rfloor}.$$

Proof. Let $P = \sum_{j=0}^n (-1)^j \binom{n}{2j}$ and $Q = \sum_{j=0}^n (-1)^j \binom{n}{2j+1}$. Since $(1+i)^n = P+iQ$ and $(1-i)^n = P-iQ$, then $P = \frac{(1+i)^n + (1-i)^n}{2}$ and $Q = i^{-1} \frac{(1+i)^n - (1-i)^n}{2}$. Therefore,

$$|P| = \begin{cases} 2^{n/2}, & \text{if } n \equiv 0 \pmod{4}; \\ 2^{(n-1)/2}, & \text{if } n \equiv 1 \pmod{4}; \\ 0, & \text{if } n \equiv 2 \pmod{4}; \\ 2^{(n-1)/2}, & \text{if } n \equiv 3 \pmod{4}; \end{cases} \text{ and } |Q| = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{4}; \\ 2^{(n-1)/2}, & \text{if } n \equiv 1 \pmod{4}; \\ 2^{n/2}, & \text{if } n \equiv 2 \pmod{4}; \\ 2^{(n-1)/2}, & \text{if } n \equiv 3 \pmod{4}; \end{cases}$$

from which the lemma follows. □

Proof of the Theorem. The homology of \mathfrak{g} is the homology of the Koszul complex $(\wedge \mathfrak{g}, \partial)$.²

Put $\mathfrak{g} = v \oplus \mathfrak{z}$. Hence, we can write $\wedge \mathfrak{g} = \wedge v \otimes \wedge \mathfrak{z}$. It is straightforward to see that

$$\partial : \wedge^p v \otimes \wedge^q \mathfrak{z} \longrightarrow \wedge^{p-2} v \otimes \wedge^{q+1} \mathfrak{z}.$$

Therefore, it follows that the complex $(\wedge \mathfrak{g}, \partial)$ is the direct sum of an *even* and an *odd* subcomplex, precisely $(\wedge^{2p} v \otimes \wedge \mathfrak{z}, \partial)$ and $(\wedge^{2p+1} v \otimes \wedge \mathfrak{z}, \partial)$. Accordingly, the homology of the Koszul complex is the sum of the homologies of each of the even and odd subcomplexes.

It is well known that if $(\mathcal{C} = (C_i), \partial)$ is a finite complex of finite dimensional vector spaces, then

$$|H_*(\mathcal{C})| \geq \left| \sum_i (-1)^i \dim(C_i) \right|.$$

By applying this to each of the even and odd subcomplexes we get that

$$\begin{aligned} |H_*(\mathfrak{g})| &\geq \left| \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \dim \left(\wedge^{2j} v \otimes \wedge \mathfrak{z} \right) \right| + \left| \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \dim \left(\wedge^{2j+1} v \otimes \wedge \mathfrak{z} \right) \right| \\ &= 2^{|\mathfrak{z}|} \left| \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{|v|}{2j} \right| + 2^{|\mathfrak{z}|} \left| \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{|v|}{2j+1} \right|. \end{aligned}$$

Now the Theorem follows from the Lemma. □

Remark. The bound obtained in the Theorem is the best possible, of the form 2^t (t an integer), that can be given in general. In fact, among the few available computations there are examples that support this claim.

² $\partial(x_1 \wedge \dots \wedge x_p) = \sum_{i < j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \dots \wedge \widehat{x}_i \wedge \dots \wedge \widehat{x}_j \wedge \dots \wedge x_p.$

Example 1. Let \mathfrak{g}_2 denote the Lie algebra with basis $\{x_1, x_2, y_1, y_2, z\}$ and non-zero brackets $[z, x_i] = y_i$ for each $1 \leq i \leq 2$. It was shown in [1] that

$$b_0 = 1, \quad b_1 = 3, \quad b_2 = 6, \quad b_3 = 6, \quad b_4 = 3, \quad b_5 = 1,$$

where $b_i = |H^i(\mathfrak{g}_2)|$. Therefore, the total cohomology of \mathfrak{g}_2 is equal to 20 and our bound is $2^{2+2} = 16$.

Example 2. Let \mathfrak{g} denote the Lie algebra with basis $\{a, b, c, d, e, f, g\}$ and non-zero brackets $[a, b] = e, [b, d] = g, [c, d] = e$ and $[a, c] = f$. This Lie algebra appears as $3, 7_D$ in Seeley's classification [6]. In [3] one can explicitly find that

$$b_0 = 1, \quad b_1 = 4, \quad b_2 = 11, \quad b_3 = 14, \quad b_4 = 14, \quad b_5 = 11, \quad b_6 = 4, \quad b_7 = 1,$$

where $b_i = |H^i(\mathfrak{g})|$. Hence $|H^*(\mathfrak{g})| = 60$, while our bound is $2^{3+2} = 32$.

Example 3. We consider here a family of examples. For each $r \geq 2$ let E be an r -dimensional vector space; then $\mathfrak{g}_r = E \oplus \wedge^2 E$ is the rank r 2-step free nilpotent Lie algebra. For $e, f \in E, [e, f] = e \wedge f$, these being all the non-zero brackets.

Their homology has been computed by Sigg [7]. Using his result one can compute, for each r , the total homology for the rank r algebra. With a simple computer program written in Maple V we have done it for small r 's. On the other hand, it is clear that our bound is $2^{\binom{r}{2} + \lceil \frac{r+1}{2} \rceil}$. In the following table we show the results for $2 \leq r \leq 13$. ($t = \binom{r}{2} + \lceil \frac{r+1}{2} \rceil$)

| r | $ \mathfrak{g}_r $ | Total homology | t | 2^t |
|-----|--------------------|--|-----|--|
| 2 | 3 | 6 | 2 | 4 |
| 3 | 6 | 36 | 5 | 32 |
| 4 | 10 | 420 | 8 | 256 |
| 5 | 15 | 9800 | 13 | 8192 |
| 6 | 21 | 452760 | 18 | 262144 |
| 7 | 28 | 4.1835024×10^7 | 25 | 3.3554432×10^7 |
| 8 | 36 | 7.691667984×10^9 | 32 | 4.294967296×10^9 |
| 9 | 45 | $2.828336198688 \times 10^{12}$ | 41 | $2.19902325552 \times 10^{12}$ |
| 10 | 55 | $2.073619375892064 \times 10^{15}$ | 50 | $1.125899906842624 \times 10^{15}$ |
| 11 | 66 | $3.040584296923128384 \times 10^{18}$ | 61 | $2.305843009213693952 \times 10^{18}$ |
| 12 | 78 | $8.898500292240756664896 \times 10^{21}$ | 72 | $4.722366482869645213696 \times 10^{21}$ |
| 13 | 91 | $5.2084270468105185237918848 \times 10^{25}$ | 85 | $3.8685626227668133590597632 \times 10^{25}$ |

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