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AUSLANDER-REITEN SEQUENCES UNDER BASE FIELD EXTENSION

STANISŁAW KASJAN

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Dedicated to Professor Helmut Lenzing on the occasion of his sixtieth birthday

Abstract. We investigate the behaviour of Auslander-Reiten sequences of modules over a finite dimensional algebra over a field k under base field extension. It is proved that an Auslander-Reiten sequence splits into a direct sum of Auslander-Reiten sequences provided the extension is separable in the sense of MacLane.

1. Introduction

Throughout this paper R denotes a finite dimensional algebra over a field k and $R^{(K)} = R \otimes_k K$, where $k \subseteq K$ is a field extension. We investigate the behaviour of the functor

$$(-) \otimes_k K : \mathrm{mod}(R) \longrightarrow \mathrm{mod}(R^{(K)}) \tag{1.1}$$

on almost split sequences and irreducible maps in the category $\mathrm{mod}(R)$ of right finitely generated R-modules. The main result is Theorem 3.8 asserting that after applying the functor (1.1) to a minimal almost split map in the category $\mathrm{mod}(R)$ we obtain a direct sum of minimal almost split maps in the category $\mathrm{mod}(R^{(K)})$ if the extension $k \subseteq K$ is separable in the sense of MacLane [4], [3]. Using this result we obtain an alternative proof of the result of Jensen and Lenzing [4], that the algebra $R^{(K)}$ is of finite representation type provided R is also and the extension $k \subseteq K$ is MacLane separable.

In Section 4 we consider the following problem: determine components $\mathcal{D}$ of the Auslander-Reiten quiver of $\mathrm{mod}(R^{(K)})$ such that if $\mathcal{D}$ contains a module defined over k, then all modules in $\mathcal{D}$ are also defined over k. We prove in Proposition 4.12 that stable and semi-stable infinite components containing a cycle of irreducible maps have the above property. We also make some observations of this kind in the case when the algebra R is hereditary.

Throughout this paper given any Artin algebra A (see [1]) we denote by $\mathrm{rad}\, A$ the Jacobson radical of A and by A^{op} the algebra opposite to A. All modules considered are right finitely generated. By $\mathrm{mod}(A)$ we denote the category of right finitely generated A-modules. Given an A-module X we denote by $\mathrm{rad}\, X$ and $\mathrm{soc}\, X$ the radical and the socle of X respectively. By Γ_A we denote the Auslander-Reiten

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quiver of A, $\tau_A = DTr$, $\tau_A^- = TrD$ are the Auslander-Reiten translates in $\mathrm{mod}(A)$. By a component of Γ_A we mean a connected component of Γ_A. We often identify vertices of Γ_A with A-modules. An arrow $X \to Y$ in Γ_A has valuation (a, b) if a is the multiplicity of X as a direct summand in the domain E of the minimal right almost split morphism $E \to Y$ and b is the multiplicity of Y as a direct summand in the codomain E' of the minimal left almost split morphism $X \to E'$. For further notation related with Auslander-Reiten quiver and almost split sequences we refer to [1].

Following [6] we denote by ${}_s\Gamma_A$ (resp. ${}_l\Gamma_A$, ${}_r\Gamma_A$) the stable (resp. left stable, right stable) part of Γ_A obtained by removing τ_A-orbits of projective and injective modules (resp. projective modules, injective modules). A connected component of ${}_s\Gamma_A$ (resp. ${}_l\Gamma_A$, ${}_r\Gamma_A$) is called a stable (resp. left stable, right stable) component of Γ_A. By a semi-stable component we mean a left stable or right stable component.

If $\mathcal{C}$ is a connected subquiver of Γ_A and X and Y are vertices of $\mathcal{C}$, then the **distance** from X to Y in $\mathcal{C}$ is by the definition the minimal length $\mathrm{dist}_{\mathcal{C}}(X, Y)$ of a (non-oriented) path in $\mathcal{C}$ connecting X with Y. The distance is zero if and only if $X \cong Y$.

Let $k \subseteq K$ be an arbitrary field extension. Given an embedding $u : X \hookrightarrow Y$ of k-vector spaces we shall identify $X \otimes_k K$ with the image of $u \otimes K$. For simplicity we shall write $X^{(K)}$ instead of $X \otimes_k K$. If X has an additional structure of an algebra or a module, then $X^{(K)}$ has the structure induced by the structure of X.

2. Preliminary facts

In this section we collect some observations concerning the behaviour of the category $\mathrm{mod}(R)$ under the action of the functor (1.1). Note that this functor is exact. We shall say that an $R^{(K)}$-module X is **defined over** k provided X is isomorphic to $Y^{(K)}$ for an R-module Y.

Lemma 2.1. *If P is a projective R-module, then $P^{(K)}$ is a projective $R^{(K)}$-module. Each indecomposable projective $R^{(K)}$-module is a direct summand of $P^{(K)}$ for some projective R-module P.*

The proof is obvious. □

Lemma 2.2. (a) *For any R-modules X, Y and $i \geq 0$ the canonical homomorphism*

$$\mathrm{Ext}^i_R(X, Y) \otimes_k K \longrightarrow \mathrm{Ext}^i_{R^{(K)}}(X^{(K)}, Y^{(K)})$$

is an isomorphism. The isomorphisms are natural with respect to homomorphisms $X' \to X$, $Y \to Y'$.

(b) *For any right R-module X and left R-module Y and $i \geq 0$ the canonical homomorphism*

$$\mathrm{Tor}^i_R(X, Y) \otimes_k K \longrightarrow \mathrm{Tor}^i_{R^{(K)}}(X^{(K)}, Y^{(K)})$$

is an isomorphism. The isomorphisms are natural with respect to homomorphisms $X \to X'$, $Y \to Y'$.

Proof. (a) The statement follows easily for $X = R$ and then for X projective. If X is arbitrary, consider the projective resolution

$$\mathbf{P}_* : \ldots \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow 0$$

of X. By applying to $\mathbf{P}_*$ the functor $(-)\otimes_k K$ we obtain by Lemma 2.1 a projective resolution of $X^{(K)}$. The complexes $\mathrm{Hom}_R(\mathbf{P}_*, Y)^{(K)}$ and $\mathrm{Hom}_{R^{(K)}}(\mathbf{P}_*^{(K)}, Y^{(K)})$ are isomorphic. The assertion (a) follows since the functor $(-)\otimes_k K : \mathrm{mod}(k) \to \mathrm{mod}(K)$ is exact.

The proof of (b) is analogous. □

Lemma 2.3. $(R^{op})^{(K)} \cong (R^{(K)})^{op}$ *and for any R-module X there exists a natural isomorphism* $\mathrm{Hom}_k(X,k)\otimes_k K \cong \mathrm{Hom}_K(X^{(K)}, K)$ *of $(R^{(K)})^{op}$-modules.*

The proof is routine. □

The following corollary is a direct consequence of Lemmata 2.3 and 2.1.

Corollary 2.4. *If E is an injective R-module, then $E^{(K)}$ is an injective $R^{(K)}$-module. Each indecomposable injective $R^{(K)}$-module is a direct summand of $E^{(K)}$ for some injective R-module E.* □

Lemma 2.5. *Let X, Y be R-modules.*

(a) *If X is indecomposable, then X is a direct summand of Y if and only if $X^{(K)}$ and $Y^{(K)}$ have a common nonzero direct summand.*

(b) $X \cong Y$ *if and only if* $X^{(K)} \cong Y^{(K)}$.

(c) *If X is indecomposable and $X^{(K)} \cong X_1 \oplus X_2$ with X_1, X_2 nonzero, then X_1 and X_2 are not defined over k.*

Proof. (a) The "only" part is obvious. For the proof of the converse implication we assume that $X^{(K)}$ and $Y^{(K)}$ have a common nonzero direct summand. It follows that there exist $R^{(K)}$-homomorphisms $f : X^{(K)} \to Y^{(K)}$ and $g : Y^{(K)} \to X^{(K)}$ such that gf is a nonzero idempotent in $\mathrm{End}_{R^{(K)}}(X^{(K)})$. It follows from Lemma 2.2 that there exist $f_i \in \mathrm{Hom}_R(X,Y)$, $g_j \in \mathrm{Hom}_R(Y,X)$, $\lambda_i, \mu_j \in K$, $i = 1, ..., r$, $j = 1, ..., s$, such that $f = \sum_{i=1}^r f_i \otimes \lambda_i$ and $g = \sum_{j=1}^s g_j \otimes \mu_j$. We treat the canonical isomorphisms from Lemma 2.2 as identities. Then $gf = \sum_{i=1}^r \sum_{j=1}^s g_j f_i \otimes \lambda_i \mu_j$. Since this is a nonzero idempotent and the ring $\mathrm{End}_R(X)$ is local, it follows that for some i, j the homomorphism $g_j f_i$ is invertible in $\mathrm{End}_R(X)$. Thus X is a direct summand of Y.

Statement (b) follows directly from (a).

(c) If X_1 is defined over k, say $X_1 \cong Y^{(K)}$ for an R-module Y, then by (a) X is a direct summand of Y. Comparing the lengths of X and Y we obtain $X \cong Y$ and $X_2 = 0$, a contradiction. □

3. MacLane separable extensions and almost split sequences

A crucial role in our considerations will be played by field extensions separable in the sense of MacLane; see Definition 3.1 below. The significance of this class of extensions for the representation theory of algebras was observed in [4].

Definition 3.1 ([3, Chapter IV, Definition 3])**.** A field extension $k \subseteq K$ is **MacLane separable** if $char\, k = 0$ or $char\, k = p > 0$ and K is linearly disjoint from $k^{p^{-1}}$ over k, that is, the natural homomorphism $K \otimes_k k^{p^{-1}} \to K k^{p^{-1}}$ is bijective.

Here $k^{p^{-1}}$ is the subfield of the algebraic closure of k consisting of elements x such that $x^p \in k$. We consider K and $k^{p^{-1}}$ as subfields of the algebraic closure $\overline{K}$ of K. □

Lemma 3.2. *If $k \subseteq K$ is a MacLane separable field extension and A is a semisimple finite dimensional k-algebra, then $A^{(K)}$ is a semisimple K-algebra.*

Proof. The assertion follows from a result of Jensen and Lenzing [4] that the global dimension of a k-algebra is preserved under a MacLane separable extension of the base field k. However for convenience of the reader we present an alternative proof of semisimplicity of $A^{(K)}$ which does not use model theoretical arguments.

It is enough to prove the assertion in the case of A being a division algebra. If A is a field, then by [3, Chapter IV, Theorem 23] we know that $A^{(K)}$ has no nonzero nilpotent elements. Since $A^{(K)}$ has finite dimension over K, it follows that $A^{(K)}$ is semisimple.

If A is not a field, then by [2, Theorem 4.2.1] there exists a maximal subfield $k \subseteq L \subseteq A$ such that $L \otimes_{Z(A)} A \cong \mathbb{M}_n(L)$, the full $n \times n$ matrix algebra with coefficients in L, where $n = [A : Z(A)]$ and $Z(A)$ is the center of A. Hence

$$L \otimes_{Z(A)} A \otimes_k K \cong \mathbb{M}_n(L \otimes_k K).$$

Since $L \otimes_k K$ is semisimple, it follows that the algebra $\mathbb{M}_n(L \otimes_k K)$ has no nonzero two-sided nilpotent ideals so $A \otimes_k K$ has no nonzero two-sided nilpotent ideals. Thus $A \otimes_k K$ is a semisimple K-algebra. □

An immediate consequence of the above lemma is the following fact.

Lemma 3.3. (a) $(\operatorname{rad} R)^{(K)} \subseteq \operatorname{rad} R^{(K)}$,
(b) $(\operatorname{rad} R)^{(K)} = \operatorname{rad} R^{(K)}$ *if $k \subseteq K$ is MacLane separable extension.*

Proof. Statement (a) follows immediately since $\operatorname{rad} R$ and consequently $(\operatorname{rad} R)^{(K)}$ are nilpotent ideals in R and $R^{(K)}$ respectively. To prove (b) it is enough to observe that it follows from Lemma 3.2 that the algebra $R^{(K)}/(\operatorname{rad} R)^{(K)}$ is semisimple if $k \subseteq K$ is a MacLane separable extension. □

Remark 3.4. The assumption that $k \subseteq K$ is MacLane separable is essential. Consider the following example from [3]. Let k be a field of characteristic 2 containing an element t being not a square of an element of k. Let $K = k(\sqrt{t}) = k[S]/(S^2 - t)$. Then $K \otimes_k K$ is commutative but contains a nonzero nilpotent element $1 \otimes s - s \otimes 1$, where s is the residue class of the variable S. Hence it is not semisimple. □

Corollary 3.5. (a) *If X is an R-module, then $(\operatorname{rad} X)^{(K)} \subseteq \operatorname{rad} X^{(K)}$ and if the extension $k \subseteq K$ is MacLane separable, then $(\operatorname{rad} X)^{(K)} = \operatorname{rad} X^{(K)}$.*

(b) *If X is an R-module, then $\operatorname{soc} X^{(K)} \subseteq (\operatorname{soc} X)^{(K)}$ and if the extension $k \subseteq K$ is MacLane separable, then $\operatorname{soc} X^{(K)} = (\operatorname{soc} X)^{(K)}$.*

(c) *If $P \to X$ is a minimal projective cover of X in $\operatorname{mod}(R)$, then the induced homomorphism $P^{(K)} \to X^{(K)}$ is a minimal projective cover of $X^{(K)}$ in $\operatorname{mod}(R^{(K)})$.*

(d) *If $X \to E$ is a minimal injective envelope of X in $\operatorname{mod}(R)$, then the induced homomorphism $X^{(K)} \to E^{(K)}$ is a minimal injective envelope of $X^{(K)}$ in $\operatorname{mod}(R^{(K)})$.*

Proof. The proof of (a) follows from the well-known fact that for a finitely generated module X over a finite dimensional algebra R we have $\operatorname{rad} X = X \operatorname{rad} R$. Statement (b) follows dually whereas (c) and (d) are direct consequences of (a), (b), Lemma 2.1 and Corollary 2.4. □

It follows immediately from the above corollary that if

$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

is a minimal projective presentation of X in $\mathrm{mod}(R)$, then the induced sequence

$$P_1^{(K)} \longrightarrow P_0^{(K)} \longrightarrow X^{(K)} \longrightarrow 0$$

is a minimal projective presentation of $X^{(K)}$ in $\mathrm{mod}(R^{(K)})$. This observation together with Lemmata 2.1, 2.2 and 2.3 yields the following corollary.

Corollary 3.6. *For any X in $\mathrm{mod}(R)$ there exist isomorphisms $(\tau_R X)^{(K)} \cong \tau_{R^{(K)}} X^{(K)}$ and $(\tau_R^- X)^{(K)} \cong \tau_{R^{(K)}}^- X^{(K)}$.* □

Now we turn to the main question of this section. We shall deal with almost split sequences in $\mathrm{mod}(R)$ and apply the functor (1.1) to them. Let us precede the main theorem by the following lemma.

Lemma 3.7. *Let A and B be rings with 1 and let S be an A-B-bimodule which is simple as a left A module and as a right B-module. Given a natural number $n \in \mathbb{N}$ let A_n and B_n be the full $n \times n$ matrix rings with coefficients in A and B respectively. Consider $S_n = S \otimes_B B_n \cong A_n \otimes_A S$ as an A_n-B_n-bimodule with the obvious structure. Then for any element $s = [s_{ij}] \in S_n$ there exist invertible elements $a \in A_n$ and $b \in B_n$ such that asb is a diagonal element of S_n, that is, $asb = [s'_{ij}]$, where $s'_{ij} = 0$ for $i \neq j$.*

Proof. The proof follows from the observation that we can diagonalize the element $[s_{ij}]$ by elementary row and column transformations. □

Theorem 3.8. *Let $X \to Y$ be a minimal left (resp. right) almost split homomorphism in $\mathrm{mod}(R)$ and the extension $k \subseteq K$ be MacLane separable. Then the induced homomorphism $X^{(K)} \to Y^{(K)}$ is isomorphic to a direct sum of minimal left (resp. right) almost split homomorphisms in $\mathrm{mod}(R^{(K)})$. In particular, if*

$$0 \longrightarrow X \longrightarrow E \longrightarrow Y \longrightarrow 0$$

is an Auslander-Reiten sequence in $\mathrm{mod}(R)$ and $Y^{(K)} \cong \bigoplus_{i=1}^r Z_i$ with indecomposable modules Z_i, then the induced exact sequence

$$0 \longrightarrow X^{(K)} \longrightarrow E^{(K)} \longrightarrow Y^{(K)} \longrightarrow 0$$

is isomorphic to the direct sum of Auslander-Reiten sequences

$$0 \longrightarrow \tau_{R^{(K)}} Z_i \longrightarrow F_i \longrightarrow Z_i \longrightarrow 0$$

for $i = 1, ..., r$.

Proof. Assume first that Y is projective and not simple. Then the embedding $\mathrm{rad}\, Y \to Y$ is the minimal right almost split morphism. Thus the assertion follows by Corollary 3.5 (a). In case of a minimal left almost split morphism starting from an injective non-simple module X we proceed dually.

Now consider the case of Y non-projective and let

$$e: \quad 0 \longrightarrow \tau_R Y \longrightarrow E \longrightarrow Y \longrightarrow 0$$

be the Auslander-Reiten sequence ending at Y. Consider the induced sequence

$$e^{(K)}: \quad 0 \longrightarrow \tau_{R^{(K)}} Y^{(K)} \longrightarrow E^{(K)} \longrightarrow Y^{(K)} \longrightarrow 0.$$

It is well known (see [1, Chapter V, Proposition 2.1], [12, proof of Theorem 11.27]) that e is represented by an element of the two-sided socle S of the $\mathrm{End}_R(\tau_R Y)$-$\mathrm{End}_R(Y)$-bimodule $\mathrm{Ext}^1_R(Y, \tau_R Y)$. It follows from Corollary 3.5 (b) that $S^{(K)}$ is the two-sided socle S' of the $\mathrm{End}_{R^{(K)}}(\tau_{R^{(K)}}Y^{(K)})$-$\mathrm{End}_{R^{(K)}}(Y^{(K)})$-bimodule

$$\mathrm{Ext}^1_{R^{(K)}}(Y^{(K)}, \tau_{R^{(K)}}Y^{(K)})$$

and the sequence $e^{(K)}$ is represented by an element $\varepsilon \in S'$.

Fix a decomposition $Y^{(K)} \cong \bigoplus_{i=1}^r Z_i$ into indecomposable $R^{(K)}$-modules Z_i. It follows from Lemmata 2.1 and 2.5 that the modules Z_i are not projective. This decomposition induces a direct sum decomposition

$$\mathrm{Ext}^1_{R^{(K)}}(Y^{(K)}, \tau_{R^{(K)}}Y^{(K)}) = \bigoplus_{i,j=1}^{n} \mathrm{Ext}^1_{R^{(K)}}(Z_i, \tau_{R^{(K)}}Z_j).$$

Observe first that $S' = \bigoplus_{i,j=1}^n S_{ij}$, where $S_{ij} = S' \cap \mathrm{Ext}^1_{R^{(K)}}(Z_i, \tau_{R^{(K)}}Z_j)$ for $i, j = 1, ..., n$. This follows from the fact that $p'_j s p_i \in \mathrm{Ext}^1_{R^{(K)}}(Z_i, \tau_{R^{(K)}}Z_j) \cap S'$ for any $s \in S'$, where p_i and p'_j are the composition of the canonical maps $\bigoplus_{l=1}^n Z_l \to Z_i \to \bigoplus_{l=1}^n Z_l$ and $\bigoplus_{l=1}^n \tau_{R^{(K)}}Z_l \to \tau_{R^{(K)}}Z_j \to \bigoplus_{l=1}^n \tau_{R^{(K)}}Z_l$ respectively for $i, j = 1, ..., n$.

Next assume that $Z_i \not\cong Z_j$. We show that $S_{ij} = 0$. Take an arbitrary element η_{ij} of S_{ij} representing an exact sequence

$$0 \longrightarrow \tau_{R^{(K)}}Z_j \longrightarrow X \longrightarrow Z_i \longrightarrow 0.$$

If this sequence does not split, then there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tau_{R^{(K)}}Z_j & \longrightarrow & F_j & \longrightarrow & Z_j & \longrightarrow & 0 \\
 & & \Big\downarrow \mathrm{id} & & \Big\downarrow & & \Big\downarrow g & & \\
0 & \longrightarrow & \tau_{R^{(K)}}Z_j & \longrightarrow & X & \longrightarrow & Z_i & \longrightarrow & 0
\end{array}$$

where the first row is an Auslander-Reiten sequence. Since the homomorphism g is not an isomorphism, it follows that η_{ij} is not annihilated by $\mathrm{rad}\,\mathrm{End}_{R^{(K)}}(Y^{(K)})$ from the right and thus $\eta_{ij} \notin S_{ij}$.

Moreover it follows from [1, Proposition 2.2] that if $Z_i \cong Z_j$, then S_{ij} coincides with the two-sided socle $\overline{S}_{ij}$ of the $\mathrm{End}_{R^{(K)}}(\tau_{R^{(K)}}Z_j)$-$\mathrm{End}_{R^{(K)}}(Z_i)$-bimodule $\mathrm{Ext}^1_{R^{(K)}}(Z_i, \tau_{R^{(K)}}Z_j)$. This bimodule is simple as a left $\mathrm{End}_{R^{(K)}}(\tau_{R^{(K)}}Z_j)$-module and as a right $\mathrm{End}_{R^{(K)}}(Z_i)$- module and its nonzero elements represent almost split sequences

$$0 \longrightarrow \tau_{R^{(K)}}Z_j \longrightarrow F \longrightarrow Z_i \longrightarrow 0.$$

We claim that for any $\eta = [\eta_{ij}] \in S'$ there exist invertible elements $f \in \mathrm{End}_{R^{(K)}}(Y^{(K)})$ and $g \in \mathrm{End}_{R^{(K)}}(\tau_{R^{(K)}}Y^{(K)})$ such that if $g\eta f = [\overline{\eta}_{ij}]$, then $\overline{\eta}_{ij} = 0$ for $i \neq j$.

Since, as we have shown, $S_{ij} = 0$ if $Z_i \not\cong Z_j$, without loss of generality we may assume that $Z_1 \cong ... \cong Z_n$. Then our claim follows from Lemma 3.7.

We have just proved that the sequence

$$0 \longrightarrow \tau_{R^{(K)}}Y^{(K)} \longrightarrow E^{(K)} \longrightarrow Y^{(K)} \longrightarrow 0$$

is isomorphic to the direct sum of exact sequences

$$0 \longrightarrow \tau_{R^{(K)}}Z_i \longrightarrow F_i \longrightarrow Z_i \longrightarrow 0$$

represented by elements $\eta_{ii} \in S_{ii}$ for $i = 1, ..., n$.

Assume that some η_{ii} is zero. It means that Z_i is a direct summand of F_i and since $E^{(K)} \cong \bigoplus_{i=1}^r F_i$ it follows by Lemma 2.5(a) that Y is a direct summand of E, a contradiction.

It follows that all elements η_{ii} are nonzero and they correspond to almost split sequences in $\operatorname{mod}(R^{(K)})$ (see [12, Theorem 11.27]). $\square$

4. Applications to irreducible morphisms and Auslander-Reiten sequences

Lemma 4.1. *Assume that the field extension $k \subseteq K$ is MacLane separable. Assume that X is an R-module and $X^{(K)} \cong \bigoplus_{i=1}^{r} X_i$ is a decomposition into indecomposable direct summands. If there exists an irreducible morphism $X_i \to Z$ (resp. $Z \to X_i$) in $\operatorname{mod}(R^{(K)})$ and Z is indecomposable, then there exist an indecomposable R-module Y and an irreducible morphism $X \to Y$ (resp. $Y \to X$) such that Z is a direct summand of $Y^{(K)}$.*

Proof. If there exists an irreducible morphism $X_i \to Z$ in $\operatorname{mod}(R^{(K)})$, then X_i is not simple injective and hence by Corollaries 2.5 and 3.5(b) X is neither. Moreover it follows from Theorem 3.8 that Z is a direct summand of $E^{(K)}$, where E is a codomain of the minimal left almost split morphism $X \to E$ in $\operatorname{mod}(R)$. We take for Y an indecomposable direct summand of E such that Z is a direct summand of $Y^{(K)}$. In case of an irreducible morphism $Z \to X_i$ the proof is analogous. $\square$

Proposition 4.2. *Assume that the field extension $k \subseteq K$ is MacLane separable. Let X, Y be indecomposable R-modules and let $X^{(K)} = \bigoplus_{i=1}^{r} X_i^{n_i}, Y^{(K)} = \bigoplus_{j=1}^{s} Y_j^{m_j}$, where the modules $X_1, ..., X_r$ (resp. $Y_1, ..., Y_s$) are indecomposable and pairwise nonisomorphic.*

(1) *The following conditions are equivalent.*

(a) *There is an irreducible morphism $X \to Y$ in* $\operatorname{mod}(R)$.

(b) *For any $1 \le i \le r$ there exist $1 \le j \le s$ and an irreducible morphism $X_i \to Y_j$ in* $\operatorname{mod}(R^{(K)})$.

(c) *For any $1 \le j \le s$ there exist $1 \le i \le r$ and an irreducible morphism $X_i \to Y_j$ in* $\operatorname{mod}(R^{(K)})$.

(d) *For some $1 \le j \le s$ and $1 \le i \le r$ there is an irreducible morphism $X_i \to Y_j$ in* $\operatorname{mod}(R^{(K)})$.

(2) *If the above conditions* (a)–(d) *hold and the arrow $X \to Y$ in Γ_R has valuation (a, b), then*

$$n_i a = \sum_{j=1}^{s} m_j a_{ij}$$

for any $i = 1, ..., r$ and

$$m_j b = \sum_{i=1}^{r} n_i b_{ij}$$

for any $j = 1, ..., s$, where (a_{ij}, b_{ij}) is the valuation of the arrow $X_i \to Y_j$ in $\Gamma_{R^{(K)}}$ if such an arrow exists and $a_{ij} = b_{ij} = 0$ otherwise.

Proof. Assume that there is an irreducible map $X \to Y$ and the corresponding arrow in Γ_R has valuation (a, b). Consider the right almost split morphism

$$X^a \oplus X' \to Y$$

where X' has no direct summand isomorphic to X. By Theorem 3.8 the induced homomorphism

$$(X^{(K)})^a \oplus X'^{(K)} \to Y^{(K)}$$

splits into a direct sum of minimal right almost split morphisms $Z_j \to Y_j$ for $j = 1, ..., s$ and since each X_i is a direct summand of some Z_j the condition (b) in (1) follows. We have just shown the implication (a)⇒(b). Moreover, calculating multiplicities of X_i and Y_j in the direct sum decompositions of $(X^{(K)})^a \oplus X'^{(K)}$ and $Y^{(K)}$ we easily obtain the first equality in the statement (2). Observe that the module $X'^{(K)}$ has no direct summand isomorphic to X_i by Lemma 2.5.

The implication (d) ⇒ (a) follows from Lemma 4.1 and 2.5(a). The remaining implications in (1) as well as the second equality in (2) can be proved analogously. □

Remark 4.3. The result above as well as Theorem 3.8 has flavour similar to the results of I. Reiten and Ch. Riedtmann [8, 4.1] concerning relationships between irreducible maps and almost split sequences over an Artin algebra Λ and a skew group algebra $\Lambda * G$. It would be interesting to work out a universal framework containing these facts.

Proposition 4.4. *Assume that the field extension $k \subseteq K$ is MacLane separable. Let $\mathcal{C}$ be a connected component of Γ_R. There exists a family $\{\mathcal{D}_s\}_{s\in S}$ of connected components of $\Gamma_{R^{(K)}}$ having the following properties.*

(1) *If X belongs to $\mathcal{C}$ and $X^{(K)}$ decomposes into a direct sum of indecomposable $R^{(K)}$-modules $X_1, ..., X_r$, then for each $s \in S$ there exists $1 \le i \le r$ such that X_i belongs to $\mathcal{D}_s$. In particular the family $\{\mathcal{D}_s\}_{s\in S}$ is finite.*

(2) *For any $s \in S$ each module in $\mathcal{D}_s$ is a direct summand of $Y^{(K)}$ for some Y in $\mathcal{C}$.*

Proof. For $\{\mathcal{D}_s\}_{s\in S}$ we take the family of those connected components of $\Gamma_{R^{(K)}}$ which contain at least one direct summand of the module $Y^{(K)}$ for a module Y in $\mathcal{C}$.

(1) Let Y be a module in $\mathcal{C}$ and Y' an indecomposable direct summand of $Y^{(K)}$ lying in $\mathcal{D}_s$. We shall prove that X_i belongs to $\mathcal{D}_s$ for some $1 \le i \le r$. We proceed by induction on the distance $\mathrm{dist}_{\mathcal{C}}(X, Y)$ of X from Y in $\mathcal{C}$.

If this distance is zero, that is, $X \cong Y$, there is nothing to prove. Assume the distance is positive and there is an irreducible morphism $X \to Z$ or $Z \to X$ in $\mathrm{mod}(R)$ with $\mathrm{dist}_{\mathcal{C}}(Z, Y) < \mathrm{dist}_{\mathcal{C}}(X, Y)$. Consider the case $X \to Z$; the remaining one is analogous. Let $Z^{(K)} = \bigoplus_{j=1}^{s} Z_j$ with Z_j indecomposable. By the induction hypothesis Z_j belongs to $\mathcal{D}_s$ for some $1 \le j \le s$. By Proposition 4.2 there exists an irreducible morphism $X_i \to Z_j$ for some i and hence X_i belongs to $\mathcal{D}_s$ as we required.

The assertion (2) follows directly from Lemma 4.1. □

By Lemmata 2.1, 2.5 and Corollaries 2.4, 3.6 an indecomposable R-module X belongs to a τ_R-orbit of a projective (resp. injective) R-module if and only if each indecomposable direct summand of $X^{(K)}$ belongs to a $\tau_{R^{(K)}}$-orbit of a projective (resp. injective) $R^{(K)}$-module. Hence the Proposition 4.4 is also valid for a stable (resp. semi-stable) component $\mathcal{C}$ of Γ_R.

Proposition 4.5. *Assume that the field extension $k \subseteq K$ is MacLane separable. Let $\mathcal{C}$ be a stable* (*resp. semi-stable*) *component of Γ_R. There exists a family $\{\mathcal{D}_s\}_{s\in S}$ of stable* (*resp. semi-stable*) *components in $\Gamma_{R^{(K)}}$ having the following properties.*

(1) *If X belongs to $\mathcal{C}$ and $X^{(K)}$ decomposes into a direct sum of indecomposable $R^{(K)}$-modules $X_1, ..., X_r$, then for each $s \in S$ there exists $1 \leq i \leq r$ such that X_i belongs to $\mathcal{D}_s$. In particular the family $\{\mathcal{D}_s\}_{s\in S}$ is finite.*

(2) *For any $s \in S$ each module in $\mathcal{D}_s$ is a direct summand of $Y^{(K)}$ for some Y in $\mathcal{C}$.*

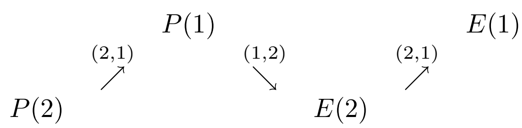
If $\mathcal{C}$ and $\{\mathcal{D}_s\}_{s\in S}$ are as Proposition 4.4 or 4.5, we denote by $\mathcal{C}^{(K)}$ the subquiver $\bigcup_{s\in S}\mathcal{D}_s$ of $\Gamma_{R^{(K)}}$.

The following result is due to Jensen and Lenzing [4] (see also [5, Proposition 12.38, Corollary 12.39]).

Corollary 4.6. *Assume that the field extension $k \subseteq K$ is MacLane separable. If R is of finite representation type, then the algebra $R^{(K)}$ is of finite representation type and each indecomposable $R^{(K)}$-module is a direct summand of a module $X^{(K)}$ for some indecomposable R-module X.*

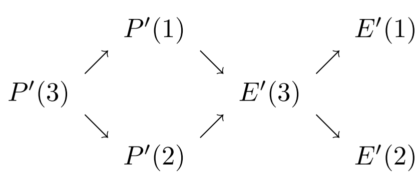
Proof. The quiver Γ_R has only finite components. It follows by Lemma 2.1 and Proposition 4.4 that each connected component of $\Gamma_{R^{(K)}}$ containing a projective $R^{(K)}$-module is finite. Thus by well-known arguments (see e.g. [1, Chapter IV, Theorem 1.4]) we see that the algebra $R^{(K)}$ is of finite representation type and each component of $\Gamma_{R^{(K)}}$ contains a projective module. The remaining statement of the corollary follows again by Proposition 4.4. □

Example 4.7. Let $R = \begin{pmatrix} \mathbb{C} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$. The Auslander-Reiten quiver Γ_R has the following form:



The valuations of the arrows are marked at the above figure. By $P(i)$ (resp. $E(i)$) we denote the indecomposable projective (resp. injective) R-module corresponding to the idempotent matrix e_i having 1 at the (i,i)-th place and zeros elsewhere.

The algebra $R^{(\mathbb{C})}$ is isomorphic to $\begin{pmatrix} \mathbb{C} & 0 & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{pmatrix}$ and the Auslander-Reiten quiver $\Gamma_{R^{\mathbb{C}}}$ has the following form:



Here $P'(i)$ (resp. $E'(i)$) denotes the i-th indecomposable projective (resp. injective) $R^{(\mathbb{C})}$-module.

Proposition 4.8. *Assume that the field extension $k \subseteq K$ is MacLane separable. If $\mathcal{C}$ is a connected component* (*resp. stable, semi-stable component*) *of Γ_R in which all arrows have trivial valuations, X is a module in $\mathcal{C}$ such that $X^{(K)}$ is indecomposable, then for each Y in $\mathcal{C}$ the induced module $Y^{(K)}$ is indecomposable. In particular $\mathcal{C}^{(K)} \cong \mathcal{C}$ as translation quivers and each module in $\mathcal{C}^{(K)}$ is defined over k.*

Proof. We proceed by induction on the distance of Y from X in $\mathcal{C}$. If the distance is zero, then $X \cong Y$ and there is nothing to prove. Let Z be a module in $\mathcal{C}$ such that $\operatorname{dist}_{\mathcal{C}}(X,Z) < \operatorname{dist}_{\mathcal{C}}(X,Y)$ and there is an irreducible morphism $Y \to Z$ or $Z \to Y$ in $\operatorname{mod}(R)$. Consider the case of $Y \to Z$; the remaining one is dual. Then by the induction hypothesis the module $Z^{(K)}$ is indecomposable. Assume that $Y^{(K)} \cong \bigoplus_{i=1}^r Y_i^{m_i}$ where the modules $Y_1, ..., Y_r$ are indecomposable and pairwise nonisomorphic. It follows from Proposition 4.2 that for any $j = 1, ..., r$ there is an arrow $Y_j \to Z^{(K)}$ in $\Gamma_{R^{(K)}}$ and if (a_j, b_j) is the valuation of this arrow, then $a = \sum_{j=1}^r m_j a_j$, where (a,b) is the valuation of the arrow $Y \to Z$ in Γ_R. Since $a = 1$ by our assumption, it follows that $r = 1$ and $m_1 = 1$ and $Y^{(K)}$ is indecomposable. □

Example 4.9. Let $k \subseteq K$ be the field extension from Remark 3.4 and let $R = k[x]/((x^2+t)^m)$ for some natural m. The Auslander-Reiten quiver of R has the form

$$k[x]/((x^2+t)^m) \quad \rightleftarrows \quad k[x]/((x^2+t)^{m-1}) \quad \rightleftarrows \quad ... \quad \rightleftarrows \quad k[x]/(x^2+t).$$

The algebra $R^{(K)}$ is isomorphic to $K[x]/(x+s)^{2m}$, where s is an element of K such that $s^2 = t$. Hence the Auslander-Reiten quiver of $R^{(K)}$ has the following form:

$$K[x]/((x+s)^{2m}) \quad \rightleftarrows \quad K[x]/((x+s)^{2m-1}) \quad \rightleftarrows \quad ... \quad \rightleftarrows \quad K[x]/(x+s).$$

Observe that not each module in this quiver is defined over k.

Lemma 4.10. *Assume that the field extension $k \subseteq K$ is MacLane separable. Let A be the path algebra of a connected quiver Q with coefficients in K and let $\mathcal{D}$ be a connected component in Γ_A. If a module X in $\mathcal{D}$ is defined over k, then all modules in $\mathcal{D}$ are defined over k.*

Proof. It is easy to check that all preprojective and preinjective modules are defined over the prime subfield of K.

Assume that $\mathcal{D}$ is regular. Let $X \cong Y^{(K)}$ be an indecomposable module in $\mathcal{D}$ and let $\mathcal{C}$ be the connected component of Γ_R containing Y. It is clear that $\mathcal{C}$ is a regular component in the Auslander-Reiten quiver of kQ and by [1, Theorem IV.4.15] either $\mathcal{D}$ is a stable tube or it is isomorphic to $\mathbb{Z}A_\infty$ (see [1, Chapter VII.4] for the notation). Then the statement follows from Proposition 4.8. □

Recall (see e.g. [10]) that an R-module is exceptional if it is indecomposable and has no selfextensions.

Corollary 4.11. *Let A be the path algebra of a quiver Q with coefficients in an algebraically closed field K and let $\mathcal{D}$ be a component in Γ_A. If $\mathcal{D}$ contains an exceptional module, then all modules in $\mathcal{D}$ are defined over the prime subfield of K.*

Proof. Without loss of generality we may assume that $\mathcal{D}$ is regular. By results of Ringel [10], see also [11], we know that an exceptional module is defined over the prime subfield F of K. Since prime fields are perfect, the extension $F \subseteq K$ is MacLane separable and our claim follows by Lemma 4.10. □

We finish with a proposition describing a class of stable and semi-stable components of $\Gamma_{R^{(K)}}$ having the similar property as connected components of the Auslander-Reiten quiver of a hereditary algebra.

Proposition 4.12. *Assume that the field extension $k \subseteq K$ is MacLane separable and assume that $\mathcal{D}$ is an infinite stable (resp. semi-stable) component in $\Gamma_{R^{(K)}}$ containing a cycle of irreducible morphisms. If there is a module X in $\mathcal{D}$ defined over k, then each module in $\mathcal{D}$ is defined over k.*

Proof. Let $X = Y^{(K)}$ for an indecomposable R-module Y and consider the stable (resp. semi-stable) component $\mathcal{C}$ of Γ_R containing Y. Then by Proposition 4.5 $\mathcal{D}$ is a connected component of $\mathcal{C}^{(K)}$. Next we note that there is a cycle of irreducible morphisms in $\mathcal{C}$. Indeed, let

$$X_1 \to X_2 \to X_3 \to ... \to X_r \to X_1$$

be a cycle in $\mathcal{D}$ and X_i is direct summand of $Y_i^{(K)}$ for an indecomposable R-module Y_i in $\mathcal{C}$. It follows by Lemma 4.1 that then there is a cycle

$$Y_1 \to Y_2 \to Y_3 \to ... \to Y_r \to Y_1$$

in $\mathcal{C}$.

Now by results of S. Liu (see [7], [6]) each arrow in $\mathcal{C}$ has trivial valuation and hence by Proposition 4.8 each module in $\mathcal{D}$ is defined over k. □

We finish the paper with the remark about the case of an algebraic separable extension $k \subseteq K$.

Proposition 4.13. *Assume that the extension $k \subseteq K$ is separable algebraic. Then any $R^{(K)}$-module is a direct summand of a module defined over k. Any connected component of the Auslander-Reiten quiver $\Gamma_{R^{(K)}}$ is a component of $\mathcal{C}^{(K)}$ for a connected component $\mathcal{C}$ of Γ_R.*

Proof. The proof follows [4]. Take an $R^{(K)}$-module Y. We shall prove that Y is a direct summand of a module defined over k. The module Y is defined over a finite extension of k; hence we can assume that the extension $k \subseteq K$ is finite separable. Then the K-K-bimodule $K \otimes_k K$ is semisimple and the trivial K-K-bimodule K is a direct summand of $K \otimes_k K$. The module Y has a natural structure of an R-module. The module $Y \otimes_k K \cong Y \otimes_K K \otimes_k K$ which is defined over k contains Y as a direct summand.

The remaining statement follows now from Proposition 4.4. □

References

[1] M. Auslander, I. Reiten and S. Smalø, *"Representation Theory of Artin Algebras"*, Cambridge Studies in Advanced Math., 36, Cambridge University Press, 1995. MR **96c:**16015

[2] I.N. Herstein, *"Noncommutative Rings"*, The Carus Mathematical Monographs, No. 15, The Mathematical Association of America, 1968. MR **37:**2790

[3] N. Jacobson, *"Lectures in Abstract Algebra"*, Vol. III, *Theory of Fields and Galois Theory*, Princeton, Van Nostrand 1964. MR **30:**3087

[4] Ch.U. Jensen and H. Lenzing, Homological dimension and representation type of algebras under base field extension, Manuscripta Math. 39 (1982), 1-13. MR **83k:**16019

[5] Ch.U. Jensen and H. Lenzing, *"Model Theoretic Algebra with particular emphasis on Fields, Rings, Modules"*, Algebra, Logic and Applications, Vol.2, Gordon & Breach Science Publishers, 1989. MR **91m:**03038

[6] S. Liu, Semi-stable components of an Auslander-Reiten quiver, *J. London Math. Soc.* (2) 47 (1993), 405-416. MR **94a:**16024

[7] S. Liu, Shapes of connected components of the Auslander-Reiten quivers of Artin algebras, in: Representation Theory of Algebras and Related Topics, Proceedings of the Workshop at UNAM, Mexico 1994, *CMS Conference Proceedings* Vol. 19 (1996), 109-137. MR **97e:**16037

[8] I. Reiten and Ch. Riedtmann, Skew group algebras in the representation theory of Artin algebras, *Journal of Algebra*, 92 (1985), 224-282. MR **86k:**16024

[9] C. M. Ringel, *"Tame Algebras and Integral Quadratic Forms"*, Lecture Notes in Math., No. 1099, Springer-Verlag, 1984. MR **87f:**16027

[10] C.M. Ringel, Exceptional modules are tree modules, SFB 343 preprint 96-082, Bielefeld 1996. CMP 98:14

[11] A. Schofield, The field of definition of a real representation of a quiver Q, *Proceedings of the Amer. Math. Soc.* 116 (1992), 293-295. MR **92m:**16016

[12] D. Simson, *"Linear Representations of Partially Ordered Sets and Vector Space Categories"*, Algebra, Logic and Applications, Vol. 4, Gordon & Breach Science Publishers, 1992. MR **95g:**16013

Faculty of Mathematics and Informatics, Nicholas Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

E-mail address: skasjan@mat.uni.torun.pl