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THE DEDEKIND-MERTENS LEMMA AND THE CONTENTS OF POLYNOMIALS

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Abstract. Let R be a commutative ring, let X be an indeterminate, and let $g \in R[X]$. There has been much recent work concerned with determining the Dedekind-Mertens number $\mu_R(g)=\min\{k \in \mathbb{N} \mid c_R(f)^{k-1}c_R(fg) = c_R(f)^k c_R(g)$ for all $f \in R[X]\}$, especially on determining when $\mu_R(g) = 1$. In this note we introduce a universal Dedekind-Mertens number $u\mu_R(g)$, which takes into account the fact that $\mu_S(g) \leq \deg(g) + 1$ for any ring S containing R as a subring, and show that $u\mu_R(g)$ behaves more predictably than $\mu_R(g)$.

Introduction

Many papers ([1], [2], [3], [4], [5], [7], [8], [9], [10], [13]) have recently considered questions concerning the following well-known result which is usually called the Dedekind-Mertens Lemma:

Lemma 0.1. *If $g \in R[X]$ and deg(g) = n, then*

$$c_R(f)^n c_R(fg) = c_R(f)^{n+1} c_R(g) \text{ for all } f \in R[X]. \tag{0.1}$$

Much of this work has been on determining the smallest n for which (0.1) holds. To obtain a refinement of Lemma 0.1, in [10] the authors defined the *Dedekind-Mertens number* $\mu_R(g)$ of $g \in R[X]$ to be the smallest positive integer k such that $c_R(f)^{k-1}c_R(fg) = c_R(f)^k c_R(g)$ for all $f \in R[X]$. Thus Lemma 0.1 states that $\mu_R(g) \leq \deg(g) + 1$. It follows that $\mu_R(g) = \sup\{\mu_{R_m}(g) \mid m$ is a maximal ideal of $R\}$. Thus in considering $\mu_R(g)$ we may as well assume R is quasilocal. In this case, the main result of [10] improves Lemma 0.1 to

$$\mu_R(g) \leq \mu_R(c_R(g)) \tag{0.2}$$

where $\mu_R(M)$ denotes the minimal number of generators of the R-module M. In [10], [4], the question of the opposite inequality to (0.2) was also considered. The special case of whether $\mu_R(g) = 1$ implies $c_R(g)$ is principal was considered as early as [14], and several further results have recently been obtained on this case ([2], [3], [4], [5], [7], [8], [9], [10]).

An important further property of the exponent n in Lemma 0.1, is that it is universal in the sense that the formula (0.1) continues to hold if f is chosen to

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have coefficients in any ring S containing R as a subring, whereas we may have $\mu_R(g) < \mu_S(g)$ [9, Remark 1.7], or $\mu_R(c_R(g)) > \mu_S(c_S(g))$. Some of the history of the Dedekind-Mertens Lemma is discussed in [6] where the importance of this independence of the base ring is stressed. The object of this note is to introduce the *universal Dedekind-Mertens number*, and to point out that if one switches from the Dedekind-Mertens number as defined above, to the universal Dedekind-Mertens number, then the counterparts to the questions considered in [10], [4] become much simpler.

1. Strong Dedekind-Mertins Lemma

The original Dedekind-Mertens Lemma, as given for example in [11], [12], [13] and [6, p. 3] is stronger than Lemma 0.1. To explain this we extend the definition of content. If $\phi : R \to S$ is a homomorphism of rings and $f \in S[X]$, we define $c_R(f)$ to be the R-submodule of S generated by the coefficients of f. If A, B are R-submodules of S, we may define AB to be the R-submodule of S generated by $\{ab \mid a \in A, b \in B\}$. Then $c_R(f)^n c_R(fg)$ and $c_R(f)^{n+1}c_R(g)$ make sense. In particular, if ϕ is an inclusion of a subring R into S, it is clear that the smaller that one chooses R the stronger the condition $c_R(f)^n c_R(fg) = c_R(f)^{n+1}c_R(g)$ becomes. The Dedekind-Mertens Lemma as given for example in [13], [1] states:

Lemma 1.1. *Let $g \in R[X]$ with $deg(g) = n$. Then*

$$c_{\mathbb{Z}}(f)^n c_{\mathbb{Z}}(fg) = c_{\mathbb{Z}}(f)^{n+1} c_{\mathbb{Z}}(g) \text{ for all } f \in R[X].$$

In particular, unlike the inequality (0.2) this condition is universal in the sense that it continues to hold if R is replaced with any ring containing the coefficients of f and g. Thus we may as well take the coefficients of f to be independent indeterminates. Because of the importance of this universality in Kronecker's use of the content to develop his theory of divisors [6], and other reasons it is of interest to have a universal version of the inequality (0.2) and other results as well. For example if A is a subring of B, $f \in A[X]$ with $c_A(f) = A$ and $g \in B[X]$ with $fg \in A[X]$, it is immediate from Lemma 1.1 but not from Lemma 0.1 that $g \in A[X]$.

If M is an R-module and $g \in M[X]$, let $c_{R\,M}(g)$ denote the R-submodule of M generated by the coefficients of g. Observe that if M is a submodule of an R-module N, then $c_{R\,M}(g) = c_{R\,N}(g)$. Thus we may just write $c_R(g)$. We will let R be a fixed quasilocal ring throughout. In considering the relationship between $\mu_R(g)$ and $\mu_R(c_R(g))$ for $g \in R[X]$ in [4], the authors defined the *polarized Dedekind-Mertens number* $\widetilde{\mu}_R(g)$ of g with respect to R to be the smallest positive integer k such that

$$\sum_{i=1}^{k} c_R(f_i g) c_R(f_1) c_R(f_2) \cdots \widehat{c_R(f_i)} \cdots c_R(f_k) = c_R(f_1) c_R(f_2) \cdots c_R(f_k) c_R(g)$$

for all $f_1, \ldots, f_k \in R$. We also define the *universal polarized Dedekind-Mertens number,* and show that it is the same as the universal Dedekind-Mertens number.

Definition 1.2. Let M be an R-module and let $g \in M[X]$. Let $T = \{t_i \mid i \in \mathbb{N}\}$ be a new set of indeterminates. The *universal Dedekind-Mertens number* $u\mu_R(g)$ of g with respect to R is the smallest positive integer k such that, for each $f \in R[T][X]$, it holds that

$$c_R(f)^{k-1} c_R(fg) = c_R(f)^k c_R(g) \text{ as submodules of } M[T] = M \otimes_R R[T].$$

The *universal polarized Dedekind-Mertens number* $u\widetilde{\mu}_R(g)$ of g with respect to R is the smallest positive integer k such that for all $f_1, \ldots, f_k \in R[T][X]$ we have

$$\sum_{i=1}^{k} c_R(f_i g) c_R(f_1) c_R(f_2) \cdots \widehat{c_R(f_i)} \cdots c_R(f_k) = c_R(f_1) c_R(f_2) \cdots c_R(f_k) c_R(g)$$

as submodules of $M[T] = M \otimes_R R[T]$.

It is clear that $u\mu_R(g) \leq u\widetilde{\mu}_R(g)$ and the proof of the Dedekind-Mertens Lemma given in [13] actually shows that $u\widetilde{\mu}_R(g) \leq \deg(g) + 1$. Also, if R is a subring of S, it follows that $u\mu_R(g) \geq u\mu_S(g)$ and $u\widetilde{\mu}_R(g) \geq u\widetilde{\mu}_S(g)$.

2. Results

While the focus of much of [2], [3], [4], [5], [7], [8], [9], [10] has been on determining the Dedekind-Mertins number $\mu_R(g)$, with many interesting partial results, if we switch to the *universal* Dedekind-Mertins number $u\mu_R(g)$, we have the following result.

Theorem 2.1. *Let (R, m) be a quasi-local ring, let M be an R-module and let $g \in M[X]$. Then $\mu_R(c_R(g)) = u\widetilde{\mu}_R(g) = u\mu_R(g)$.*

Proof. As noted before we have $u\mu_R(g) \leq u\widetilde{\mu}_R(g)$. The proof of the opposite inequality is similar to that of [4, Lemma 2.4]. Let $u\mu_R(g) = k$ and let $f_1, \ldots, f_k \in R[T][X]$. Let $N_1 > \max\{\deg_X(f_1), \deg_X(f_1 g)\}$, and if $N_1, \ldots, N_{i-1}$ have been defined, let $N_i > \max\{\deg_X(f_i), \deg_X(f_i g) + N_{i-1}\}$. Let $t_1, \ldots, t_{k-1}$ be members of T which do not appear in any of the f_i, and let $T' = T - \{t_1, \ldots, t_{k-1}\}$.

By the definition of $u\mu_R(g)$ we have

$$\begin{aligned} c_R(f_1 + \sum_{i=2}^{k} f_i t_{i-1} X^{N_{i-1}})^{k-1} c_R([f_1 + \sum_{i=2}^{k} f_i t_{i-1} X^{N_{i-1}}] g) \\ = c_R(f_1 + \sum_{i=2}^{k} f_i t_{i-1} X^{N_{i-1}})^{k} c_R(g). \end{aligned}$$

By the choice of the N_i this is

$$\begin{aligned} [c_R(f_1) + \sum_{i=2}^{k} c_R(f_i) t_{i-1}]^{k-1} [c_R(f_1 g) + \sum_{i=2}^{k} c_R(f_i g) t_{i-1}] \\ = [c_R(f_1) + \sum_{i=2}^{k} c_R(f_i) t_{i-1}]^{k} c_R(g). \end{aligned}$$

Considering these as polynomials in $t_1, \ldots, t_{k-1}$ with coefficients in $R[X, T']$, and comparing the coefficients of the monomial $t_1 t_2 \cdots t_{k-1}$, we get

$$\sum_{i=1}^{k} c_R(f_i g) c_R(f_1) c_R(f_2) \cdots \widehat{c_R(f_i)} \cdots c_R(f_k) = c_R(f_1) c_R(f_2) \cdots c_R(f_k) c_R(g).$$

Thus $u\mu_R(g) \geq u\widetilde{\mu}_R(g)$.

The proof that $u\mu_R(g) \leq \mu_R(c_R(g))$ is very similar to the proof given in [10] that $\mu_R(g) \leq \mu_R(c_R(g))$.

Lemma 2.2. *Let (R, m) be a quasilocal ring, let T be a countably infinite set of independent indeterminates over $R[X]$, let M be an R-module and let $g \in M[X]$. Let $b \in mc_R(g)$ and $h = g + bX^i$. Let A be a finitely generated R-submodule of $R[T]$ and let $f \in R[T][X]$. If $Ac_R(f)c_R(h) = Ac_R(fh)$, then $Ac_R(f)c_R(g) = Ac_R(fg)$.*

Proof. It suffices to show $Ac_R(f)c_R(g) \subseteq Ac_R(fg)$. Since $b \in mc_R(g)$ and $h = g + bX^i$, we have $c_R(h) \subseteq c_R(g) \subseteq c_R(h) + mc_R(g)$, and thus $c_R(g) = c_R(h)$ by Nakayama's Lemma. Thus

$$Ac_R(f)c_R(g) = Ac_R(f)c_R(h) = Ac_R(fh) = Ac_R(f(g + bX^i))$$

$$\subseteq Ac_R(fg) + Abc_R(f) \subseteq Ac_R(fg) + mAc_R(f)c_R(g).$$

By Nakayama's Lemma we have $Ac_R(f)c_R(g) = Ac_R(fg)$. □

To show $u\mu_R(g) \leq \mu_R(c_R(g))$, we may assume that $c_R(g)$ is minimally generated by $k \geq 2$ elements, and that if $h \in M[X]$ with $c_R(h)$ minimally generated by fewer than k elements, then for any $f \in R[T][X]$ we have $c_R(f)^{k-2}c_R(fh) = c_R(f)^{k-1}c_R(h)$.

Let $g = b_mX^m + \cdots + b_1X + b_0$. By the above lemma we may assume b_m is a minimal generator of $c_R(g)$. Then $g = b_mh + g_1$, where $c_R(g_1)$ is generated by fewer than k elements and $h \in R[X]$ with $c_R(h) = R$.

Write $f = a_nX^n + f_1$ where $\deg(f_1) < \deg(f) = n$. By induction on $\deg(f)$ we may also assume $c_R(f_1)^{k-1}c_R(fg) = c_R(f_1)^k c_R(g)$.

Claim 1. $c_R(fg_1) \subseteq c_R(fg) + b_mc_R(f_1)$.

Indeed we have

$$c_R(fg_1) = c_R(f(g - b_mh)) \subseteq c_R(fg) + c_R(b_mhf) = c_R(fg) + b_mc_R(f)$$

$$= c_R(fg) + b_mc_R(a_nX^n + f_1) \subseteq c_R(fg) + a_nb_mR + b_mc_R(f_1),$$

and since $a_nb_m \in c_R(fg)$, this is $c_R(fg) + b_mc_R(f_1)$. This proves Claim 1.

Claim 2. $c_R(f_1g) \subseteq c_R(fg) + a_nc_R(g_1)$.

Indeed we have

$$c_R(f_1g) = c_R((f - a_nX^n)g) \subseteq c_R(fg) + a_nc_R(g) \subseteq c_R(fg) + a_nc_R(b_mh + g_1)$$

$$\subseteq c_R(fg) + a_nb_mR + a_nc_R(g_1) = c_R(fg) + a_nc_R(g_1).$$

The last equality holds since $a_nb_m \in c_R(fg)$. This proves Claim 2.

Now to prove $u\mu_R(g) \leq \mu_R(c_R(g)) = k$, it suffices to show that each term of $c_R(f)^kc_R(g)$ of the form $\theta = a_0^{v_0}a_1^{v_1}\cdots a_n^{v_n}b_j$, with $\sum v_i = k$, is in $c_R(f)^{k-1}c_R(fg)$.

Case 1. If $v_n \neq 0$ and $j = m$, then $\theta = a_0^{v_0}a_1^{v_1}\cdots a_n^{v_n}b_m \in c_R(f)^{k-1}c_R(fg)$.

Case 2. If $v_n \neq 0$ and $j < m$, then $b_j = b_me_j + b_{1j}$, where e_j is a coefficient of h and b_{1j} is a coefficient of g_1, and $\theta = a_0^{v_0}a_1^{v_1}\cdots a_n^{v_n}b_j = a_0^{v_0}a_1^{v_1}\cdots a_n^{v_n}(b_me_j + b_{1j}) \in c_R(f)^{k-1}c_R(fg) + c_R(f)^{k-1}a_nc_R(g_1)$.

Case 3. If $v_n = 0$, then $\theta = a_0^{v_0}a_1^{v_1}\cdots a_{n-1}^{v_{n-1}}b_j \in c_R(f_1)^kc_R(g) = c_R(f_1)^{k-1}c_R(f_1g)$ by the induction hypothesis on the degree of f.

Combining the three cases we have

$$c_R(f)^kc_R(g) \subseteq c_R(f)^{k-1}c_R(fg) + c_R(f)^{k-1}a_nc_R(g_1) + c_R(f_1)^{k-1}c_R(f_1g).$$

Applying Claim 2 to the third term on the right, we see that this is contained in

$$c_R(f)^{k-1}c_R(fg) + c_R(f)^{k-1}a_nc_R(g_1) + c_R(f_1)^{k-1}(c_R(fg) + a_nc_R(g_1)).$$

Since $c_R(f_1) \subseteq c_R(f)$ the product $c_R(f_1)^{k-1}(c_R(fg) + a_n c_R(g_1))$ is contained the other two terms. Thus

$$c_R(f)^k c_R(g) \subseteq c_R(f)^{k-1} c_R(fg) + c_R(f)^{k-1} a_n c_R(g_1). \tag{2.1}$$

Now since $c_R(g_1)$ is generated by fewer than k elements, we have $c_R(f)^{k-2}c_R(fg_1) = c_R(f)^{k-1}c_R(g_1)$ by induction on k. Thus the right side of (2.1) is

$$c_R(f)^{k-1}c_R(fg) + a_n c_R(f)^{k-2}c_R(fg_1).$$

By Claim 1, this is contained in

$$c_R(f)^{k-1}c_R(fg) + a_n c_R(f)^{k-2}(c_R(fg) + b_m c_R(f_1))$$

$$= c_R(f)^{k-1}c_R(fg) + a_n c_R(f)^{k-2}c_R(fg) + a_n b_m c_R(f)^{k-2}c_R(f_1).$$

But $a_n c_R(f)^{k-2}c_R(fg) \subseteq c_R(f)^{k-1}c_R(fg)$ and

$$a_n b_m c_R(f)^{k-2}c_R(f_1) = (c_R(f)^{k-2}c_R(f_1))(a_n b_m) \subseteq c_R(f)^{k-1}c_R(fg).$$

Thus $c_R(f)^k c_R(g) = c_R(f)^{k-1}c_R(fg)$, showing that $u\mu_R(g) \leq \mu_R(c_R(g))$.

It remains to prove $\mu_R(c_R(g)) \leq u\mu_R(g)$. This will follow from the next proposition.

Proposition 2.3. *Let $g \in R[X]$ have degree m. If $\mu_R(c_R(g)) > k$, and $f \in R[T][X]$ has independent indeterminates for its coefficients, and deg(f) $= n > mk - k^2$, then $c_R(f)^{k-1}c_R(fg) \neq c_R(f)^k c_R(g)$. Thus $u\mu_R(g) > k$.*

Proof. Let $\mu_R(c_R(g)) = k + j$, $j \geq 1$. Then $\mu_R(c_R(f)) = n + 1$, and $\mu_R(c_R(f)^k) = \binom{n+k}{n}$ = the number of monomials of degree k in $n+1$ variables. Also the inequality $mk - k^2 < n$ is easily seen to be equivalent to $\binom{n+k-1}{n}(m+n+1) < \binom{n+k}{n}(k+1)$. Then

$$\mu_R(c_R(f)^{k-1}c_R(fg)) \leq \binom{n+k-1}{n}(m+n+1)$$

$$< \binom{n+k}{n}(k+1) \leq \binom{n+k}{n}(k+j) = \mu_R(c_R(f)^k c_R(g)).$$

Thus if $\mu_R(c_R(g)) \geq k+1$, then $u\mu_R(g) \geq k+1$. □

We state two interesting special cases of the above proposition.

(i) If $\mu_R(c_R(g)) > 1$ and $f \in R[T][X]$ has independent indeterminates for its coefficients, and $\deg(f) = n > m - 1$, then $c_R(fg) \neq c_R(f)c_R(g)$.

(ii) If $\mu_R(c_R(g)) \geq m$ and $f \in R[T][X]$ has independent indeterminates for its coefficients, and $\deg(f) = n > m^2 - m^2 = 0$, then $c_R(f)^{m-1}c_R(fg) \neq c_R(f)^m c_R(g)$.

Remark 2.4. Let M be an R-module and let $g \in M[X]$. The *universal Dedekind-Mertens number* $u\mu_R(g)$ of g with respect to R is also the smallest positive integer k such that, for each commutative R-algebra S and each $f \in S[X]$, it holds that

$$c_R(f)^{k-1}c_R(fg) = c_R(f)^k c_R(g) \text{ as submodules of } M \otimes_R S.$$

Similarly the *universal polarized Dedekind-Mertens number* $u\widetilde{\mu}_R(g)$ of g with respect to R is the smallest positive integer k such that for each commutative R-algebra S

and for all $f_1, \ldots, f_k \in S[X]$ we have

$$\sum_{i=1}^{k} c_R(f_i g) c_R(f_1) c_R(f_2) \cdots \widehat{c_R(f_i)} \cdots c_R(f_k) = c_R(f_1) c_R(f_2) \cdots c_R(f_k) c_R(g)$$

as submodules of $M \otimes_R S$.

Proof. Let k be the smallest positive integer such that for each commutative R-algebra S and each $f \in S[X]$, it holds that

$$c_R(f)^{k-1} c_R(fg) = c_R(f)^k c_R(g) \text{ as submodules of } M \otimes_R S.$$

By taking $S = R[T]$ where T is a countably infinite set of indeterminates over $R[X]$, we get $k \geq u\mu_R(g)$.

Also, if S is a commutative R-algebra and $f = a_m X^m + \cdots + a_0 \in S[X]$, let $h = t_m X^m + \cdots + t_0 \in R[T][X]$, where the $t_i \in T$ are distinct indeterminates. Choose any R-algebra homomorphism $\sigma : R[T][X] \to S[X]$ such that $\sigma(X) = X$ and $\sigma(t_i) = a_i$ for $i = 0, \ldots, m$. Then σ induces a homomorphism $M \otimes_R R[T][X] \to M \otimes_R S[X]$ which, for $n = u\mu_R(g)$, carries $c_R(h)^{n-1} c_R(hg) = c_R(h)^n c_R(g)$ to $c_R(f)^{n-1} c_R(fg) = c_R(f)^n c_R(g)$. Thus $k \leq u\mu_R(g)$. □

□

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