

## DISTINCT SUBSET SUMS AND AN INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. In this note we prove an inequality for convex functions which implies a conjecture of P. Erdős about a finite integer set with distinct subset sums.

Let  $a_0 < a_1 < \cdots < a_n$  be positive integers with sums  $\sum_{i=0}^n \epsilon_i a_i$  ( $\epsilon_i = 0, 1$ ) different. P. Erdős conjectured that

$$\sum_{i=0}^n \frac{1}{a_i} \leq \sum_{i=0}^n \frac{1}{2^i}.$$

This conjecture was proved by Ryavec (see [1]). Several different proofs are given in [4] by A. Bruen and D. Borwein, O.P. Lössers, M. Edelstein and Esther Szekeres. Hanson, Steele and Stenger [3] proved that

$$\sum_{i=0}^n \frac{1}{a_i^\alpha} \leq \sum_{i=0}^n \frac{1}{2^{\alpha i}}$$

for all  $\alpha > 0$ . Frenkel [2] further improved this by proving

$$(1) \quad \sum_{i=0}^n f(a_i) \leq \sum_{i=0}^n f(2^i)$$

for any convex decreasing function. In this note we prove an inequality for convex functions. (1) is a special case of the inequality.

**Theorem.** *Let  $f$  be any given convex decreasing function on  $[A, B]$  and  $\alpha_0, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n$  real numbers in  $[A, B]$  with*

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n, \quad \sum_{i=0}^k \alpha_i \geq \sum_{i=0}^k \beta_i, \quad k = 0, 1, \dots, n.$$

*Then*

$$\sum_{i=0}^n f(\alpha_i) \leq \sum_{i=0}^n f(\beta_i).$$

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*Proof.* We transform  $\alpha_0, \alpha_1, \dots, \alpha_n$  into  $\alpha'_0, \alpha'_1, \dots, \alpha'_n$ , and then use induction on  $n$ . The case  $n = 0$  is clear. Let

$$\delta = \min \left\{ \frac{1}{n}(B - \alpha_n), \frac{1}{j+1} \left( \sum_{i=0}^j \alpha_i - \sum_{i=0}^j \beta_i \right) : j = 0, 1, \dots, n-1 \right\}$$

and  $\alpha'_i = \alpha_i - \delta$  ( $i = 0, 1, \dots, n-1$ ),  $\alpha'_n = \alpha_n + n\delta$ . Then  $\alpha'_0, \alpha'_1, \dots, \alpha'_n$  satisfies the condition of the theorem, and either  $\alpha'_n = B$  (in this case we say that  $j_0 = n-1$ ) or  $\sum_{i=0}^{j_0} \alpha'_i = \sum_{i=0}^{j_0} \beta_i$  for some  $j_0$ ,  $0 \leq j_0 \leq n-1$ . Thus both  $\alpha'_0, \alpha'_1, \dots, \alpha'_{j_0}; \beta_0, \beta_1, \dots, \beta_{j_0}$  and  $\alpha'_{j_0+1}, \dots, \alpha'_n; \beta_{j_0+1}, \dots, \beta_n$  satisfy the condition of the theorem. By the induction hypothesis we have

$$\sum_{i=0}^{j_0} f(\alpha'_i) \leq \sum_{i=0}^{j_0} f(\beta_i), \quad \sum_{i=j_0+1}^n f(\alpha'_i) \leq \sum_{i=j_0+1}^n f(\beta_i).$$

Hence

$$\sum_{i=0}^n f(\alpha'_i) \leq \sum_{i=0}^n f(\beta_i).$$

Further,

$$\sum_{i=0}^n f(\alpha_i) \leq \sum_{i=0}^n f(\alpha'_i) = f(\alpha_0 - \delta) + \dots + f(\alpha_{n-1} - \delta) + f(\alpha_n + n\delta)$$

follows from the property of the convex function

$$f(\alpha - \gamma) + f(\beta + \gamma) \geq f(\alpha) + f(\beta)$$

where  $A \leq \alpha - \gamma \leq \alpha \leq \beta \leq \beta + \gamma \leq B$ . This completes the proof.  $\square$

*Remark.* By taking  $g(x) = -f(x)$ ,  $-f(B-x)$  and  $f(B-x)$  we may derive similar inequalities for concave increasing, concave decreasing and convex increasing functions respectively.

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