PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 11, Pages 3317–3327
S 0002-9939(00)05405-8
Article electronically published on May 11, 2000

ON THE EXISTENCE OF DOUBLING MEASURES WITH CERTAIN REGULARITY PROPERTIES

PER BYLUND AND JAUME GUDAYOL

(Communicated by Dale Alspach)

ABSTRACT. Given a compact pseudo-metric space, we associate to it upper and lower dimensions, depending only on the pseudo-metric. Then we construct a doubling measure for which the measure of a dilated ball is closely related to these dimensions.

1. Introduction

Let (X, ρ) be a compact metric space. Suppose that (X, ρ) is homogeneous. This means that there exists a doubling measure μ supported by X; i.e. there is a constant c such that, for $x \in X$ and $R > 0$, one has $0 < \mu(B(x, R)) < \infty$ and

$$\mu(B(x, 2R)) \leq c\mu(B(x, R)). \tag{1}$$

Dynkin proved in [Dyn] that for certain subsets E of the unit sphere $\mathbb{T} \subset \mathbb{C}$ there exists a doubling measure on E, and he conjectured that any compact $E \subset \mathbb{R}^n$ is homogeneous. This conjecture was proved in [V-K] by using a dimension first defined in [Lar] called the uniform metric dimension, in this paper denoted by $\Upsilon(E)$. More precisely, Volberg and Konyagin proved that (X, ρ) is homogeneous if and only if there is some $s < \infty$ such that any ball $B(x, kR)$ contains at most Ck^s points separated from each other by a distance of at least R. The uniform metric dimension $\Upsilon(X) = \Upsilon(X, \rho)$ is then defined as the infimum of such s. Furthermore, given $s < \infty$ in the condition above Volberg and Konyagin proved that for any $s' > s$ there exists a measure μ such that, for $0 < R \leq kR$,

$$\mu(B(x, kR)) \leq Ck^{s'}\mu(B(x, R)). \tag{2}$$

Clearly, any measure satisfying (2) is a doubling measure, and conversely, iterating (1) one gets (2) with $s' = \log_2 c$. In particular, Volberg and Konyagin proved Dynkin's conjecture by showing that on any compact $E \subset \mathbb{R}^n$ there exists a measure μ satisfying (2) with $s = n$ (in the maximum metric).

In this paper we generalize their result by showing the existence of a measure μ not only satisfying (2), but also the following analogous lower bound condition. Suppose there is a $t \geq 0$ such that any ball $B(x, kR)$ contains at least Ck^t points

Received by the editors May 14, 1998 and, in revised form, January 4, 1999.
2000 *Mathematics Subject Classification.* Primary 28C15; Secondary 54E45, 54F45.
The second author is partially supported by MEC grant PB95-0956-C02-01 and CIRIT grant GRQ94-2014.

separated from each other by a distance of at least R. Then for any $t' < t$ there exists a measure μ such that, for $0 < R \leq kR$,

$$Ck^{t'}\mu(B(x,R)) \leq \mu(B(x,kR)). \tag{3}$$

In [J-W] Jonsson and Wallin studied function spaces on subsets of $\mathbb{R}^n$ supporting measures fulfilling both (2) and (3) in the special case $s' = t'$. Such sets are also called s-sets or Ahlfors-regular sets.

The general case $t' \leq s'$ was considered in [Jon].

The authors of this paper, independently of each other, also studied the general case $t' \leq s'$ in [Byl] and [Gud]. Each of these works contains the main result of this paper, in [Byl] formulated for Euclidean spaces and in [Gud] for metric spaces.

In this paper the result is stated in terms of pseudo-metric spaces.

2. Definitions and statements of results

Throughout we denote by $X = (X, d)$ a compact pseudo-metric space, where $d : X \times X \mapsto [0, +\infty)$ is a pseudo-metric on X, i.e.

1. $d(x,y) = 0 \iff x = y$,
2. $d(x,y) = d(y,x), \quad \forall x, y \in X$,
3. there is a constant C_d such that $d(x,z) \leq C_d(d(x,y) + d(y,z)), \ \ \forall x, y, z \in X$.

On X we consider the topology generated by the open pseudo-balls, and without loss of generality we assume that $\operatorname{diam}(X) < 1$.

Given any ball $B(x, kR)$, $x \in X$ and $0 < R \leq kR$, denote by $N(x, R, k)$ the maximum number of points in $B(x, kR)$ separated by a distance greater than or equal to R from each other.

Definition 1. Define $X \in \Upsilon_s$ if there exists $C = C(s)$ such that, for $0 < R \leq kR$,

$$N(x, R, k) \leq Ck^s. \tag{Υ_s}$$

The upper dimension $\Upsilon(X)$ is then defined as

$$\Upsilon(X) = \inf\{s \mid X \in \Upsilon_s\}.$$

$\Upsilon(X)$ was introduced in [Lar] called the uniform metric dimension.

Definition 2. A positive Borel measure $\mu \in U_s$ if there exists $C = C(s)$ such that, for $x \in X$ and $0 < R \leq kR$,

$$\mu(B(x,kR)) \leq Ck^s\mu(B(x,R)). \tag{U_s}$$

The dimension $U(X)$ is then defined as

$$U(X) = \inf\{s \mid U_s \neq \emptyset\}.$$

Note that by taking $k = 1/R$ in (U_s) one gets the weaker condition

$$\mu(B(x,R)) \geq CR^s, \qquad x \in X, \quad 0 < R. \tag{U_s'}$$

Also note that if $\mu \in U_s$, for some s, then $\operatorname{supp}(\mu) = X$. As mentioned in the introduction, μ is doubling precisely when $\mu \in U_s$ for some $s < \infty$. We will write $\mathcal{U} = \bigcup_s U_s$ for the set of all doubling measures on X.

Volberg and Konyagin proved ([V-K]) that $\Upsilon(X) \leq U(X)$, and furthermore:

Theorem 1 (Volberg-Konyagin). *Let X be a compact metric space. If $X \in \Upsilon_s$, then for any $s' > s$ there exists a measure $\mu \in U_{s'}$. Consequently, $\Upsilon(X) = U(X)$.*

Our main result is Theorem 2 extending Theorem 1 to the analogue lower dimension. Note that Theorem 2 is stated for pseudo-metric spaces.

We start by defining the concept of the lower dimension.

The lower dimension.

Definition 3. Define $X \in \Lambda_t$ if there exists $C = C(t)$ such that, for $x \in X$ and $0 < R \le kR$,

$$N(x, R, k) \ge Ck^t. \qquad (\Lambda_t)$$

The lower dimension $\Lambda(X)$ is then defined as

$$\Lambda(X) = \sup\{t \mid X \in \Lambda_t\}.$$

$\Lambda(X)$ was introduced in ([Lar]) called the minimal dimension. Note that $X \in \Lambda_0$ is trivial.

Definition 4. A positive Borel measure $\mu \in L_t$ if there exists $C = C(t)$ such that, for $x \in X$ and $0 < R \le kR$,

$$\mu(B(x, kR)) \ge Ck^t \mu(B(x, R)). \qquad (L_t)$$

As before, by taking $k = 1/R$ in (L_t) one gets the weaker condition

$$\mu(B(x, R)) \le CR^t, \qquad x \in X, \quad 0 < R. \qquad (L'_t)$$

Now, observe that defining the lower dimension as

$$L(X) = \sup\{t \mid L_t \ne \emptyset\}$$

will not work since $\mu \in L_t$ does not imply $\mathrm{supp}(\mu) = X$, so this will say nothing about $X \setminus \mathrm{supp}(\mu)$. The appropriate definition is as follows.

Definition 5. Define the lower dimension $L(X)$ as

$$L(X) = \sup\{t \mid L_t \cap \mathcal{U} \ne \emptyset\}.$$

Note that L_0 poses no restriction on $\mu \in \mathcal{U}$.

The main theorem. We now state the main result of this paper. Note that in the special case $t = 0$ one can take $t' = t = 0$.

Theorem 2. *Let $X \in \Upsilon_s \cap \Lambda_t$, $0 \le t \le s < +\infty$, be a compact pseudo-metric space. Then for any $s' > s$ and $t' < t$ there is a probability measure $\mu \in U_{s'} \cap L_{t'}$.*

From Theorem 2 and Propositions 4 and 5 below we then get

Corollary 3. *If $\Upsilon(X) < +\infty$, then $\Upsilon(X) = U(X)$ and $\Lambda(X) = L(X)$.*

3. Proof of the theorem

To prove Theorem 2 we construct a sequence of measures with certain properties and the desired measure μ will be a limit point of this sequence.

We start by proving the trivial inequalities $\Upsilon(X) \le U(X)$ and $\Lambda(X) \ge L(X)$.

3.1. **The trivial inequalities.**

Proposition 4. *If $\mu \in U_s$, then $X \in \Upsilon_s$, i.e.* $\Upsilon(X) \leq U(X)$.

Proof. Let $\mu \in U_s$, fix any $x \in X$ and let $x_1, \dots, x_N$ be points in $B(x, kR)$ with $d(x_i, x_j) \geq R$ for $i \neq j$. Since $\mu \in U_s$ and $B(x, 2C_d kR) \subset B(x_i, 4C_d^2 kR)$,

$$\mu(B(x, 2C_d kR)) \leq \mu(B(x_i, 4C_d^2 kR)) \leq C8^s C_d^{3s} k^s \mu(B(x_i, \frac{R}{2C_d})).$$

Also, the balls $B(x_i, R/(2C_d))$ are disjoint and lie in $B(x, 2C_d kR)$, so

$$\mu(B(x, 2C_d kR)) \geq \sum_{i=1}^{N} \mu(B(x_i, \frac{R}{2C_d})) \geq N \frac{\mu(B(x, 2C_d kR))}{C8^s C_d^{3s} k^s}.$$

Thus $N \leq Ck^s$, i.e. $\Upsilon(X) \leq U(X)$. □

Proposition 5. *If $\mu \in L_t \cap \mathcal{U}$, then $X \in \Lambda_t$, i.e.* $\Lambda(X) \leq L(X)$.

Proof. Let $\{x_1, \dots, x_N\}$ be a maximal set of points in $B(x, kR)$ separated by a distance greater than or equal to R. Fix any $\mu \in L_t \cap \mathcal{U}$. Then, since μ is doubling and $B(x_i, kR) \subset B(x, 2C_d kR)$ for all i,

$$C\mu(B(x, kR)) \geq \mu(B(x, 2C_d kR)) \geq \mu(B(x_i, kR)) \geq Ck^t \mu(B(x_i, R)).$$

Also, $B(x, kR) \subset \bigcup_{i=1}^{N} B(x_i, R)$, since $\{x_1, \dots, x_N\}$ is maximal, i.e.

$$\mu(B(x, kR)) \leq \sum_{i=1}^{N} \mu(B(x_i, R)) \leq \frac{N}{Ck^t} \mu(B(x, kR)).$$

Thus, $N \geq Ck^t$, i.e. $X \in \Lambda_t$. □

3.2. **The main lemma.** Assume that $X \in \Lambda_t \cap \Upsilon_s$. Let C_d be the constant associated to the pseudo-metric d, C_t the constant appearing in Λ_t and C_s the one in Υ_s. Given $t' < t$ and $s' > s$, choose $A \geq 16C_d^4$ large enough such that $A^{s'-s} > C_s$ and $A^{t-t'} > 4^t C_d^{2t} C_t^{-1}$. For each non-negative integer j, let S_j be a maximal set of points in X separated by a distance greater than or equal to A^{-j}.

Define mappings $\mathcal{E} = \mathcal{E}_m : S_{m+1} \to S_m$ for $m \geq 0$ as follows. For $g \in S_{m+1}$ choose one of the points $e \in S_m$ for which $d(g, e) = d(g, S_m)$, and denote it by $e = \mathcal{E}(g)$. Then for $e \in S_m$ let

$$S_{e,m+1} = \{g \in S_{m+1},\ e = \mathcal{E}(g)\}.$$

It is easy to see that $\{S_{e,m+1} \mid e \in S_m\}$ form a partition of S_{m+1}.

The desired measure μ will be a limit of measures μ_m supported by S_m. Lemma 7 below will allow us to perform the inductive step that constructs μ_{m+1} from μ_m. First though we need the following preparatory lemma.

Lemma 6. *Let $e \in S_m$. Then*

$$A^{t'} \leq \#(S_{e,m+1}) \leq A^{s'},$$

where # denotes the cardinality of a set.

Proof. Fix any $e \in S_m$. Clearly $S_{e,m+1} \subset B(e, A^{-m})$ since S_m is maximal. Therefore, and since $X \in \Upsilon_s$ and $A^{s'-s} > C_s$,

$$\#(S_{e,m+1}) \leq \#(S_{m+1} \cap B(e, A^{-m})) \leq N(e, A^{-m-1}, A) \leq C_s A^s \leq A^{s'},$$

which proves the right inequality of the lemma.

For the left inequality, we first note that there exists $g \in S_{m+1}$ for which $d(g,e) < A^{-m-1}$, and as $A > 2C_d$ it is clear that $e = \mathcal{E}(g)$ for such g.

Also, for $e' \neq e''$ we have $B(e', A^{-m}/2C_d) \cap B(e'', A^{-m}/2C_d) = \emptyset$. Thus,

$$(*) \qquad S_{m+1} \cap B(e, A^{-m}/(2C_d)) \subset S_{e,m+1}.$$

Next, for $\{g_i\}_{i=1}^n = S_{m+1} \cap B(e, A^{-m}/2C_d)$ we have

$$(\dagger) \qquad n \geq N(e, A^{-m-1}, A/2C_d^2 - 1).$$

To check it, suppose the contrary, that is, suppose that

$$n < N(e, A^{-m-1}, A/2C_d^2 - 1) = n_1.$$

Then there would exist points $x_1, \dots, x_{n_1}$ in $B(e, (A/2C_d^2 - 1)A^{-m-1})$ separated from each other by a distance greater than or equal to A^{-m-1}.

But, for $g \in S_{m+1} \setminus (S_{m+1} \cap B(e, A^{-m}/2C_d))$ we have

$$d(g, x_i) \geq \frac{1}{C_d} d(g,e) - d(e, x_i) \geq \frac{A}{2C_d^2} A^{-m-1} - \left(\frac{A}{2C_d^2} - 1\right) A^{-m-1} = A^{-m-1},$$

which means that the set

$$S'_{m+1} = (\{x_i\}_{i=1}^{n_1} \cup S_{m+1}) \setminus (S_{m+1} \cap B(e, \frac{A}{2C_d} A^{-m-1}))$$

fulfills $\#(S'_{m+1}) > \#(S_{m+1})$, a contradiction to the maximality of S_{m+1}.

Thus, from $(*)$, $(\dagger)$, the choice of A and the fact that $X \in \Lambda_t$, we conclude

$$\begin{aligned} \#(S_{e,m+1}) &\geq \#(S_{m+1} \cap B(e, A^{-m}/2C_d)) \geq N(e, A^{-m-1}, A/2C_d^2 - 1) \\ &\geq C_t \left(A/2C_d^2 - 1\right)^t \geq C_t A^t (4C_d^2)^{-t} \geq A^{t'}. \end{aligned}$$

□

Lemma 7. *Let f_0 be a measure on S_m such that for any $e, e' \in S_m$ we have*

$$f_0(e') \leq C_1 f_0(e)$$

whenever $d(e,e') \leq C_2 A^{-m}$, with $C_1 = A^{s'-t'}$, and $C_2 = 8C_d^3$. Then there is a measure f_1 on S_{m+1} with the following properties:

- **(a)** *$f_1(g') \leq C_1 f_1(g)$ for any $g, g' \in S_{m+1}$ with $d(g,g') \leq C_2 A^{-m-1}$.*
- **(b)** *If $g \in S_{e,m+1}$, then $A^{-s'} f_0(e) \leq f_1(g) \leq A^{-t'} f_0(e)$.*
- **(c)** *$f_0(X) = f_1(X)$.*
- **(d)** *The construction of the measure f_1 from the measure f_0 can be regarded as a transfer of mass from the points in S_m to those of S_{m+1}, with no mass transferred over a distance greater than $2C_d A^{-m}$. This means that if $g \in S_{m+1}$ receives mass from $e \in S_m$, then $d(g,e) \leq 2C_d A^{-m}$.*

Proof of the lemma. Let f_{00} be the measure obtained by homogeneously distributing the mass of each $e \in S_m$ on the points in $S_{e,m+1}$. By doing so, we obtain a measure satisfying **(b)** (because of Lemma 6), **(c)** and **(d)**. If f_{00} satisfies **(a)**, then let $f_1 = f_{00}$ and we are done.

Assume that f_{00} does not satisfy **(a)**. Let $\{g'_i, g''_i\}_{i=1}^T$ be all the pairs of points in S_{m+1} with $d(g'_i, g''_i) \leq C_2 A^{-m-1}$. We will construct a finite sequence of measures $\{f_{0j},\ j = 1, \dots, T\}$, such that f_{0j} will satisfy **(a)** for all the pairs $\{(g'_i, g''_i)\}_{i=1}^j$, and as we will see $f_1 = f_{0T}$ is the desired measure.

The construction of f_{0j+1} from f_{0j} is as follows:

If $C_1^{-1} f_{0j}(g''_{j+1}) \le f_{0j}(g'_{j+1}) \le C_1 f_{0j}(g''_{j+1})$, then let $f_{0j+1} = f_{0j}$. Otherwise, only one of these inequalities can fail, and without loss of generality we may assume that $f_{0j}(g'_{j+1}) > C_1 f_{0j}(g''_{j+1})$. Then we move mass from g'_{j+1} to g''_{j+1} by defining f_{0j+1} as

$$\begin{aligned} f_{0j+1}(g'_{j+1}) &= f_{0j}(g'_{j+1}) - \frac{f_{0j}(g'_{j+1}) - C_1 f_{0j}(g''_{j+1})}{C_1+1}; \\ f_{0j+1}(g''_{j+1}) &= f_{0j}(g''_{j+1}) + \frac{f_{0j}(g'_{j+1}) - C_1 f_{0j}(g''_{j+1})}{C_1+1}; \\ f_{0j+1}(g) &= f_{0j}(g) \qquad \text{if } g \notin \{g'_{j+1}, g''_{j+1}\}. \end{aligned}$$

With this definition $f_{0j+1}(g'_{j+1}) = C_1 f_{0j+1}(g''_{j+1})$, which means that **(a)** is true for f_{0j+1} with respect to (g'_{j+1}, g''_{j+1}). In particular, note that **(a)** is true for f_{01} with respect to (g'_1, g''_1).

We are now going to check condition **(b)** for f_{0j+1}. To do so, suppose that **(b)** holds for f_{0j}, i.e. suppose that

$$A^{-s'} f_0(e) \le f_1(g) \le A^{-t'} f_0(e), \quad g \in S_{e,m+1}.$$

If $f_{0j+1} = f_{0j}$ or $g \notin \{g'_{j+1}, g''_{j+1}\}$, then there is nothing to check. Otherwise, as before we can assume that $f_{0j}(g'_{j+1}) > C_1 f_{0j}(g''_{j+1})$. Let $e' = \mathcal{E}(g'_{j+1})$ and $e'' = \mathcal{E}(g''_{j+1})$. It is clearly enough to prove that $f_{0j+1}(g'_{j+1}) \ge A^{-s'} f_0(e')$ and $f_{0j+1}(g''_{j+1}) \le A^{-t'} f_0(e'')$ (because $f_{0j+1}(g'_{j+1}) < f_{0j}(g'_{j+1}) \le A^{-t'} f_0(e')$ and $f_{0j+1}(g''_{j+1}) > f_{0j}(g''_{j+1}) \ge A^{-s'} f_0(e'')$). Now

$$\begin{aligned} d(e', e'') &\le C_d d(e', g'_{j+1}) + C_d^2 d(g'_{j+1}, g''_{j+1}) + C_d^2 d(g''_{j+1}, e'') \\ &\le C_d A^{-m} + C_2 C_d^2 A^{-m-1} + C_d^2 A^{-m} \le C_2 A^{-m}, \end{aligned}$$

so $f_0(e') \le C_1 f_0(e'')$. Therefore

$$\begin{aligned} f_{0j+1}(g''_{j+1}) &= C_1^{-1} f_{0j+1}(g'_{j+1}) \le C_1^{-1} f_{0j}(g'_{j+1}) \\ &\le C_1^{-1} A^{-t'} f_0(e') \le A^{-t'} f_0(e''). \end{aligned}$$

Analogously, $f_0(e'') \ge C_1^{-1} f_0(e')$. Thus,

$$\begin{aligned} f_{0j+1}(g'_{j+1}) &= C_1 f_{0j+1}(g''_{j+1}) \ge C_1 f_{0j}(g''_{j+1}) \\ &\ge C_1 A^{-s'} f_0(e'') \ge A^{-s'} f_0(e'). \end{aligned}$$

Consequently, since **(b)** holds for f_{00} it is clear that it holds for $f_1 = f_{0T}$ as well.

We are now going to check that when a pair satisfies **(a)** with respect to f_{0j}, it also does with respect to f_{0j+1}. To this end, pick any pair (g_1, g_2), $d(g_1, g_2) \le C_2 A^{-m-1}$, for which

$$C_1^{-1} f_{0j}(g_1) \le f_{0j}(g_2) \le C_1 f_{0j}(g_1).$$

If (g_1, g_2) and (g'_{j+1}, g''_{j+1}) have no point in common or if $f_{0j+1} = f_{0j}$, then we are done. Otherwise, $f_{0j+1} \ne f_{0j}$ and $f_{0j}(g'_{j+1}) > C_1 f_{0j}(g''_{j+1})$. Then the two pairs have only one point in common, say g_1. In this case $f_{0j+1}(g_2) = f_{0j}(g_2)$.

We have two possible cases to consider, either $g_1 = g'_{j+1}$ or $g_1 = g''_{j+1}$:

1. If $g_1 = g''_{j+1}$, then $f_{0j+1}(g_1) > f_{0j}(g_1)$. Thus, in this case it is enough to prove that $f_{0j+1}(g_1) \le C_1 f_{0j+1}(g_2)$. Let $e' = \mathcal{E}(g'_{j+1})$ and $e_2 = \mathcal{E}(g_2)$. Then

$$\begin{aligned} d(e', e_2) &\le C_d d(e', g'_{j+1}) + C_d^3 d(g'_{j+1}, g_1) + C_d^3 d(g_1, g_2) + C_d^2 d(g_2, e_2) \\ &\le C_d A^{-m} + 2C_d^3 C_2 A^{-m-1} + C_d^2 A^{-m} \le C_2 A^{-m}, \end{aligned} \tag{4}$$

so $f_0(e') \le C_1 f_0(e_2)$. Also, since we already know that **(b)** is true, we have $f_0(e_2) \le A^{s'} f_{0j+1}(g_2)$ and $f_{0j+1}(g'_{j+1}) \le A^{-t'} f_0(e')$. Thus,

$$\begin{aligned} f_{0j+1}(g_1) &= f_{0j+1}(g''_{j+1}) = C_1^{-1} f_{0j+1}(g'_{j+1}) \le C_1^{-1} A^{-t'} f_0(e') \\ &\le A^{-t'} f_0(e_2) \le A^{s'-t'} f_{0j}(g_2) = A^{s'-t'} f_{0j+1}(g_2) = C_1 f_{0j+1}(g_2). \end{aligned}$$

2. Otherwise, if $g_1 = g'_{j+1}$, then $f_{0j+1}(g_1) < f_{0j}(g_1)$. Thus, it is enough to check that $f_{0j+1}(g_1) \ge C_1^{-1} f_{0j+1}(g_2)$. But, for $e'' = \mathcal{E}(g''_{j+1})$, then as in (4), $d(e'', e_2) \le C_2 A^{-m}$. Also, $f_{0j+1}(g_1) = C_1 f_{0j+1}(g''_{j+1})$. Thus, from **(b)** we then get

$$\begin{aligned} f_{0j+1}(g_1) &= C_1 f_{0j+1}(g''_{j+1}) \ge C_1 A^{-s'} f_0(e'') \ge A^{-s'} f_0(e_2) \\ &\ge A^{t'-s'} f_{0j+1}(g_2) = C_1^{-1} f_{0j+1}(g_2). \end{aligned}$$

This concludes the proof that **(a)** is true for f_1.

Clearly $f_{0j+1}(X) = f_{0j}(X)$, so **(c)** is also true for f_1.

It remains to check **(d)**. When passing from f_0 to f_{00} no mass is moved over a distance exceeding A^{-m}, because $S_{e,m+1} \subset B(e, A^{-m})$, and when going from f_{0j} to f_{0j+1} no mass is moved over a distance exceeding $C_2 A^{-m-1}$, and $C_2/A < 1$. It therefore remains to prove that in the construction of f_1 from f_0 there are no pairs (g_1, g_2) and (g_2, g_3) in S_{m+1} for which mass is first moved from g_1 to g_2 and then at a subsequent step from g_2 to g_3. To prove this, assume the opposite. Then

$$f_{00}(g_1) > C_1 f_{00}(g_2) \qquad \text{and} \qquad f_{00}(g_2) > C_1 f_{00}(g_3).$$

But, if $e_1 = \mathcal{E}(g_1)$ and $e_3 = \mathcal{E}(g_3)$, then as in (4), $d(e_1, e_3) \le C_2 A^{-m}$, so by the hypothesis $C_1^{-1} f_0(e_1) \le f_0(e_3) \le C_1 f_0(e_1)$. Also,

$$A^{-s'} f_0(e_i) \le f_{00}(g_i) \le A^{-t'} f_0(e_i),$$

for $i = 1$ and $i = 3$. Adding these two inequalities, we would then get

$$f_0(e_1) \ge A^{t'} f_{00}(g_1) > C_1 A^{t'} f_{00}(g_2) > C_1^2 A^{t'} f_{00}(g_3) \ge C_1^2 A^{t'-s'} f_0(e_3),$$

contradicting $f_0(e_1) \le C_1 f_0(e_3)$, as $d(e_1, e_3) \le C_2 A^{-m}$ and $C_1 = A^{s'-t'}$. □

3.3. **Proof of the theorem.** We will now use Lemma 7 to construct a sequence of probability measures and prove that any limit point of this sequence belongs to $L_{t'} \cap U_{s'}$.

We start by defining a probability measure μ_0 on S_0 (note that S_0 consists of one point only, by the assumption $\operatorname{diam}(X) < 1$). Obviously μ_0 satisfies the hypothesis of Lemma 7. By using Lemma 7 to construct $\mu_{j+1} = f_1$ on S_{j+1} from $\mu_j = f_0$, $j \ge 0$, we then get a sequence $\{\mu_j\}_{j=0}^{\infty}$ of probability measures. This sequence belongs to the unit ball of the dual of the Banach space $\mathcal{C}(X)$, and thus has at least one weak limit point. Let μ be any limit point of this sequence. In the proof we will frequently use the following proposition, based on **(d)** of Lemma 7.

Proposition 8. *Let $j \in \mathbb{N}$, $r \geq 0$, $x \in X$ and put $C_3 = 2C_d^2/(1 - C_d/A)$. Then*

$$\mu_j(B(x,r)) \leq \mu(B(x, r + C_3 A^{-j}))$$

and

$$\mu(B(x,r)) \leq \mu_j(B(x, r + C_3 A^{-j})).$$

Proof. According to **(d)** of Lemma 7 no mass is moved at a distance exceeding $2C_d A^{-j}$ when constructing μ_{j+1} from μ_j. Thus, when passing from μ_j to μ_{j+k}, $k \geq 1$, no mass is moved at a distance exceeding

$$2C_d^2 A^{-j} \sum_{n=0}^{k-1} (C_d/A)^n < \frac{2C_d^2}{1 - C_d/A} A^{-j} = C_3 A^{-j},$$

which means that there is no mass transfer from $B(x,r)$ into the complement of $B(x, r + C_3 A^{-j})$, and vice versa. Thus,

$$\mu_j(B(x,R)) \leq \mu_{j+k}(B(x, r + C_3 A^{-j}))$$

and

$$\mu_{j+k}(B(x,r)) \leq \mu_j(B(x, r + C_3 A^{-j})).$$

Now, as μ is a weak limit point of $\{\mu_{j+k}\}$, the same is true for μ as well. □

We will now prove that $\mu \in L_{t'} \cap U_{s'}$. To this end, fix $x \in X$ and some R and k for which $0 < R \leq kR$. Then choose integers m and M such that

$$kR \leq A^{-m} < AkR \qquad \text{and} \qquad \frac{R}{A} \leq A^{-M} < R. \tag{5}$$

Denote by e_{M+1} one of the points in S_{M+1} closest to x (there may be several) and for $j = 0, \ldots, M - m$ define $e_{M-j} = \mathcal{E}(e_{M-j+1}) \in S_{M-j}$.

First claim.

$$\mu_{m+2}(e_{m+2}) \leq \mu(B(x, kR)) \leq C_s 3^{s'} (1 + C_3)^s C_1 \mu_m(e_m). \tag{6}$$

Proof. By the definition of e_{M-j} and property 3 of the pseudo-metric d, we have

$$d(x, e_{m+2}) \leq C_d A^{-m-2} \sum_{j=0}^{\infty} (C_d/A)^j = \frac{C_d}{1 - C_d/A} A^{-m-2}.$$

Let $y \in B(e_{m+2}, C_3 A^{-m-2})$. Then, by (5),

$$d(y, x) \leq C_d C_3 A^{-m-2} + \frac{C_d^2}{1 - C_d/A} A^{-m-2} \leq A^{-m-1} < kR,$$

i.e. $B(e_{m+2}, C_3 A^{-m-2}) \subset B(x, kR)$. From Proposition 8 we then get

$$\mu_{m+2}(e_{m+2}) \leq \mu(B(e_{m+2}, C_3 A^{-m-2})) \leq \mu(B(x, kR)),$$

proving the left inequality in (6). To prove the right inequality, note that (5) and Proposition 8 imply

$$\mu(B(x, kR)) \leq \mu_m(B(x, kR + C_3 A^{-m})) \leq \mu_m(B(x, (1 + C_3) A^{-m})).$$

But, $d(x, e_m) \leq \frac{C_d}{1 - C_d/A} A^{-m}$. Thus, if $e \in S_m \cap B(x, (1 + C_3) A^{-m})$, then

$$d(e, e_m) \leq C_d (1 + C_3) A^{-m} + \frac{C_d^2}{1 - C_d/A} A^{-m} \leq C_2 A^{-m},$$

so from Lemma 7 it follows that $\mu_m(e) \le C_1\mu_m(e_m)$. Now,

$$\#\left(S_m \cap B(x,(1+C_3)A^{-m})\right) \le C_s(1+C_3)^s,$$

so from Proposition 8 and the fact that $kR \le A^{-m}$, we get

$$\mu(B(x,kR)) \le \mu_m(B(x,(1+C_3)A^{-m})) \le C_s(1+C_3)^s C_1\mu_m(e_{x,m}),$$

which concludes the proof of the first claim.

Second claim.

$$\mu_{M+1}(e_{M+1}) \le \mu(B(x,R)) \le C_s(1+C_3)^{s'}A^{2s'}C_1\mu_{M+1}(e_{M+1}). \tag{7}$$

Proof. By the definition of e_{M+1},

$$d(e_{M+1},x) = d(x,S_{M+1}) \le A^{-M-1} < R/A.$$

Thus, for $y \in B(e_{x,M+1}, C_3A^{-M-1})$,

$$d(y,x) \le C_dC_3A^{-M-1} + C_dA^{-M-1} \le A^{-M} < R.$$

Again by Proposition 8,

$$\mu_{M+1}(e_{M+1}) \le \mu(B(e_{M+1},C_3A^{-M-1})) \le \mu(B(x,R)),$$

proving the left inequality in (7). To prove the right inequality, note that from Proposition 8 and the fact that $R \le A^{-M+1}$, by the choice of M,

$$\mu(B(x,R)) \le \mu_{M-1}(B(x,R+C_3A^{-M+1})) \le \mu_{M-1}(B(x,(1+C_3)A^{-M+1})).$$

Also, for $g \in B(x,R+C_3A^{-M+1}) \cap S_{M-1}$,

$$\begin{aligned} d(g,e_{M-1}) &\le C_dd(g,x) + C_d^3d(x,e_{M+1}) + C_d^3d(e_{M+1},e_M) + C_d^2d(e_M,e_{M-1}) \\ &\le C_d(1+C_3)A^{-M+1} + C_d^3A^{-M-1} + C_d^3A^{-M} + C_d^2A^{-M+1} \le C_2A^{-M+1}. \end{aligned}$$

Thus, from **(a)** and **(b)** of Lemma 7 we get (recalling $e_{M-j} = \mathcal{E}(e_{M-j+1})$),

$$\mu_{M-1}(g) \le C_1\mu_{M-1}(e_{M-1}) \le C_1A^{2s'}\mu_{M+1}(e_{M+1}).$$

But,

$$\#\left(B(x,(1+C_3)A^{-M+1}) \cap S_{M-1}\right) \le C_s(1+C_3)^s,$$

so

$$\mu(B(x,R)) \le \mu_{M-1}(B(x,(1+C_3)A^{-M+1})) \le C_s(1+C_3)^sA^{2s'}C_1\mu_{M+1}(e_{M+1}),$$

proving the second claim. To conclude the proof, note that

$$\mu(e_m) \le A^{s'(M+1-m)}\mu_{M+1}(e_{M+1}) \quad \text{and} \quad \mu_{m+2}(e_{m+2}) \ge A^{t'(M-m-1)}\mu_{M+1}(e_{M+1})$$

by **(b)** in Lemma 7. Also note that $k < A^{M-m} \le A^2k$, by the choice of m and M.

Thus, from the two claims it follows that

$$\mu(B(x,kR)) \le C\mu_m(e_m) \le CA^{s'(M-m)}\mu_{M+1}(e_{M+1}) \le Ck^{s'}\mu(B(x,R)),$$

and similarly,

$$\mu(B(x,kR)) \ge \mu_{m+2}(e_{m+2}) \ge CA^{t'(M-m)}\mu_{M+1}(e_{M+1}) \ge Ck^{t'}\mu(B(x,R)),$$

i.e. $\mu \in \Lambda_{t'} \cap \Upsilon_{s'}$.

Note that the final constants C depend only on the given constants C_d, C_s, C_t and the choice of A, s' and t'. Also note that the last inequality depends on the fact that $\Upsilon(X) < +\infty$. □

4. The non-compact case

In [L-S] Theorem 1 was generalized to a non-compact complete metric space X. It is easy to see that their proof holds for a pseudo-metric, too. We conclude this paper by showing that Theorem 2 combined with their proof gives the analogue generalization of Theorem 2 as well. Before that we just briefly sketch their proof, and refer to [L-S] for details:

Let $s' > s$ and cover $X \in \Upsilon_s$ with a countable collection of compact balls $X_n = B(x_0, n)$, $n \in \mathbb{N}$, $x_0 \in X$. Every X_n carries a $\mu_n \in U_{s'}$, by Theorem 1.

By using the weak-* compactness of the unit ball of $\mathcal{C}(X_n)$ and a Cantor's diagonal process they show the existence of a subsequence $\{\mu_j^*\}$ of $\{\mu_n\}$ such that, for every continuous $f \geq 0$ with compact support on X_p, $\int_{X_p} f d\mu_j^*$ converges to $\int_X f d\mu$ for some $\mu \in U_{s'}$.

Theorem 9. *Let $X \in \Upsilon_s \cap \Lambda_t$ be any complete pseudo-metric space. Then there exists a $\mu \in U_{s'} \cap L_{t'}$ for every $t' < t$ and $s' > s$.*

Proof. We use the notation above. It remains to prove $\mu \in L_{t'}$. From Theorem 2 it is clear that $\mu_j^* \in U_{s'} \cap L_{t'}$, where the constant C in $L_{t'}$ is the same for all j. Let $x \in X$, $r > 0$, $k > 1$. Let $0 < \varepsilon < (k-1)/(k+1)$ and pick continuous functions $0 \leq f, g \leq 1$ such that $f = 1$ on $B(x, (1-\varepsilon)kr)$ and $g = 1$ on $B(x, r)$, and such that f and g have compact support on $B(x, kr)$ and $B(x, (1+\varepsilon)r)$, respectively. Put $c^{-1} = Ck^{t'}((1-\varepsilon)/(1+\varepsilon))^{t'}$, choose p such that $B(x, kr) \subset X_p$ and choose j large enough that $|\int_{X_p} h d\mu - \int_{X_p} h d\mu_j^*| < \varepsilon$ for $h = f, g$. Then

$$\begin{aligned}\mu(B(x,r)) \leq \int_{X_p} g d\mu \leq \int_{X_p} g d\mu_j^* + \varepsilon \leq \mu_j^*(B(x,(1+\varepsilon)r)) + \varepsilon \\ \leq c\mu_j^*(B(x,(1-\varepsilon)kr)) + \varepsilon \leq c\int_{X_p} f d\mu_j^* + \varepsilon \\ \leq c\int_{X_p} f d\mu + c\varepsilon + \varepsilon \leq c\mu(B(x,kr)) + c\varepsilon + \varepsilon\end{aligned}$$

Letting $\varepsilon \to 0$ gives $Ck^{t'}\mu(B(x,r) \leq \mu(B(x,kr)$, i.e. $\mu \in L_{t'}$. □

References

[Byl] Per Bylund, *Besov spaces and measures on arbitrary closed sets*, Doctoral Thesis No 8, 1994, Department of Mathematics, University of Umeå, Sweden. MR **95k:**46046

[Dyn] E. M. Dynkin, *Free interpolation by functions with derivatives in H^1*, J. Soviet Math. **27** (1984), 2475-2481.

[Gud] Jaume Gudayol, *Boundary behaviour of functions in Hardy-Sobolev spaces*, doctoral dissertation, Universitat de Barcelona, 1997.

[Jon] Alf Jonsson, *Besov spaces on closed subsets of $\mathbb{R}^n$*, Transactions of the AMS **41** (1994), no. 1, 355-370. MR **94c:**46065

[J-W] Alf Jonsson and Hans Wallin, *Function spaces on subsets of $\mathbb{R}^n$*, Math. Reports Volume 2, Part 1, Harwood Academics Publ. GmbH, 1984.

[Lar] D.G. Larman, *A new theory of dimension*, Proc. London Math. Soc. (3rd series) **17** (1967), 178-192. MR **34:**3540

[L-S] J. Luukkainen and E. Saksman, *Every complete doubling metric space carries a doubling measure*, Proceedings of the AMS **126** (1998), 531-534. MR **99c:**28009

[V-K] A.L. Vol'berg and S.V. Konyagin, *On measures with the doubling condition*, Math. USSR Izvestiya **30** (1988), 629-638. MR **88i:**28006

Department of Mathematics, University of Umeå, S-90187 Umeå, Sweden
E-mail address: Per.Bylund@math.umu.se

Departament de Matèmatica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain
E-mail address: gudayol@mat.ub.es