

SUPERPOSITION OPERATOR IN SOBOLEV SPACES ON DOMAINS

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ABSTRACT. For an arbitrary open set $\Omega \subset \mathbb{R}^n$ we characterize all functions G on the real line such that $G \circ u \in W^{1,p}(\Omega)$ for all $u \in W^{1,p}(\Omega)$. New element in the proof is based on Maz'ya's capacity criterion for the imbedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Let Ω be an open set in \mathbb{R}^n , $n \geq 2$, and let $W^{1,p}(\Omega)$ be a Sobolev space with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}, \quad 1 \leq p < \infty.$$

Let $G : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ be a Borel function. Then the associated superposition operator T_G is given by

$$u \mapsto G \circ u.$$

The purpose of the present paper is to find the necessary and sufficient conditions on G for

$$(1) \quad T_G : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega).$$

This and more general problems were considered for $p \neq n$ and domains Ω with Lipschitz boundaries in [9], [10], and for $\Omega = \mathbb{R}^n$, $1 \leq p < \infty$, in [3]. Following [3] we say that space $W^{1,p}(\Omega)$ is supercritical if $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, and subcritical if $W^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega)$. By $|E|$ we denote Lebesgue measure of the set $E \subset \mathbb{R}^n$. Let $B(x, r)$ denote an open ball in \mathbb{R}^n with center x and radius r .

The core of the works [10], [3] is the criterion for (1) when Ω is a cube. If Ω is a cube, then (1) is equivalent to $G' \in L^\infty(\mathbb{R}^1)$ for subcritical $W^{1,p}(\Omega)$ (that is, for $1 \leq p \leq n$), and to $G' \in L^\infty_{\text{loc}}(\mathbb{R}^1)$ for supercritical $W^{1,p}(\Omega)$ (that is, for $p > n$). As a direct consequence of this result we obtain that for an arbitrary open set $\Omega \subset \mathbb{R}^n$

$$(2) \quad T_G : W^{1,p}_{\text{loc}}(\Omega) \rightarrow W^{1,p}_{\text{loc}}(\Omega)$$

is equivalent to $G' \in L^\infty(\mathbb{R}^1)$ if $1 \leq p \leq n$, and to $G' \in L^\infty_{\text{loc}}(\mathbb{R}^1)$ if $p > n$. Moreover, if for a domain Ω , $|\Omega| < \infty$, the exponent $p = n$, as in the Sobolev imbedding theorem, separates subcritical and supercritical $W^{1,p}(\Omega)$, then (1) is equivalent to (2). It seems that the broadest class of such domains Ω is the class

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of *John domains*, [6], [2], [5]. For domains with infinite measure one has analogous results provided the condition $G(0) = 0$ is added to the restrictions on G' , [10], [3].

For an arbitrary Ω (1) is not equivalent to (2). For general open sets Ω our main result is Theorem 1 below. For an arbitrary Ω in subcritical case with $p > n$ the condition for (2) is necessary but not sufficient for (1). On the other hand, condition $G' \in L^\infty(\mathbb{R}^1)$ is obviously sufficient for (1), but not necessary for (2) with $p > n$. In this case, in contrast with local constructions in [10], [3], we establish estimates near the boundary.

Our consideration is based on the following criterion for $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, which was proved by Maz'ya. For $x \in \Omega$ we define

$$V_\Omega(x) = \{f \in C_{\text{loc}}^\infty(\Omega) : 0 \leq f \leq 1, f(x) = 1\}.$$

Relative p -capacity in Ω of a point $x \in \Omega$ with respect to the ball $B(x, r)$ is defined (see [7]) as

$$\text{cap}_p(x, B(x, r); \Omega) = \inf \left\{ \int_\Omega |\nabla f|^p : f \in V_\Omega(x), f|_{\Omega \setminus B(x, r)} \equiv 0 \right\}.$$

Maz'ya [7] proved that $W^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega)$ if and only if

$$(3) \quad \inf \{ \text{cap}_p(x, B(x, r); \Omega) : x \in \Omega \} = 0 \quad \text{for all } r > 0.$$

Note that for $B(x, r) \subset \subset \Omega$ and $p > n$

$$(4) \quad \text{cap}_p(x, B(x, r); \Omega) = c(n, p)r^{n-p} \rightarrow \infty \quad \text{as } r \rightarrow 0$$

(see [7]).

It should be mentioned that there is a vast literature concerning characterisation of G for T_G to act in Besov and Lizorkin-Triebel spaces on \mathbb{R}^n . We refer to surveys [4], [12], [13], and to the book [11]. Results for superposition operators in other spaces of real functions, such as Lebesgue spaces, BV , and ideal spaces, can be found in [1].

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $|\Omega| < \infty$, $1 \leq p < \infty$. The following conditions on the function $G : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ are necessary and sufficient for $T_G : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)$:*

- (i) $G' \in L^\infty(\mathbb{R}^1)$, if $W^{1,p}(\Omega)$ is subcritical.
- (ii) $G' \in L_{\text{loc}}^\infty(\mathbb{R}^1)$, if $W^{1,p}(\Omega)$ is supercritical.

The proof of the theorem is given at the end of the paper. First we make some remarks.

Remark 2. Condition (3) is equivalent to $W^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega)$ also in the case $|\Omega| = \infty$. For open sets Ω of infinite measure Theorem 1 is valid provided we add the obvious necessary condition $G(0) = 0$ in both (i) and (ii). The proof is the same as below.

Remark 3. Let Ω be a *connected* open set and let ω be an open nonempty set with compact closure $\bar{\omega} \subset \Omega$. Following [7] we introduce the Banach space $L^{1,p}(\Omega)$ with the norm

$$\|u\|_{L^{1,p}(\Omega)} = \|u\|_{L^p(\omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

It is known that norms corresponding to different choices of ω are equivalent (see [7]). If $|\Omega| < \infty$ then condition (3) is equivalent to $L^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega)$ (see [7]). Theorem 1 holds for the space $L^{1,p}(\Omega)$ whenever $|\Omega| < \infty$. The proof is even easier than for $W^{1,p}(\Omega)$, as we do not need to control $\|u\|_{L^p(\Omega)}$.

Remark 4. From [9] and [10] it follows that the nonlinear operator T_G in Theorem 1 is bounded and continuous in $W^{1,p}(\Omega)$ as soon as (1) is valid. The same is true for T_G in $L^{1,p}(\Omega)$ provided $|\Omega| < \infty$, and in $W^{1,p}(\Omega)$ if $|\Omega| = \infty$.

Proof of Theorem 1. The only part of the theorem not covered by conditions from [3] for (2) is the necessity for $p > n$ in (i).

Thus we need to prove that if $T_G : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)$, $p > n$, and $W^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega)$, then $G' \in L^\infty(\mathbb{R}^1)$. From [9] and [10] it follows that G is absolutely continuous on \mathbb{R}^1 and $G' \in L^\infty_{loc}(\mathbb{R}^1)$. Moreover, for all $u \in W^{1,p}(\Omega)$ one has $\nabla(G \circ u) = G'(u)\nabla u$ a.e. in Ω .

Seeking a contradiction suppose that $G' \notin L^\infty(\mathbb{R}^1)$. Then we shall construct a function $U \in W^{1,p}(\Omega)$ such that $\|\nabla(G \circ U)\|_{L^p(\Omega)} = \infty$. To do this we first construct a special sequence of functions $\{u_j\}$, $u_j \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C_{loc}(\Omega)$ for $j = 1, 2, \dots$, and a sequence of pairwise disjoint balls $\{B(y_j, \rho_j)\} = \{B_j\}$, $B_j \subset\subset \Omega$, with the properties

$$(5) \quad \|u_j\|_{L^p(\Omega)} + \left(\int_{\Omega \setminus \cup_{m \leq j} B_m} |\nabla u_j|^p \right)^{1/p} \leq \left(1 + \dots + \frac{1}{2^j} \right) \quad \text{for all } j \geq 1,$$

$$(6) \quad u_m|_{B_j} = u_j|_{B_j} \quad \text{for } m \geq j.$$

The following inequalities are valid for all $j \geq 1$:

$$(7) \quad \int_{B_j} |\nabla u_j|^p \leq 1/j^2,$$

$$(8) \quad \int_{B_j} |G'(u_j)|^p |\nabla u_j|^p \geq C(n),$$

with some $C(n) > 0$ independent of j .

By (5)–(7) the sequence $\|u_j\|_{W^{1,p}(\Omega)}$ is bounded, and we can extract a subsequence converging weakly to some function $U \in W^{1,p}(\Omega)$. Now (6) and (8) imply that $\nabla(G \circ U) \notin L^p(\Omega)$.

We construct the sequences $\{u_j\}$ and $\{B_j\}$ by induction. Using (4) we choose a constant $R = R(n, p)$ such that for $B(z, R) \subset\subset \Omega$

$$(9) \quad \text{cap}_p(z, B(z, R); \Omega) \geq 2.$$

Without loss of generality we can assume that $G' \notin L^\infty(\mathbb{R}_+^1)$, where $\mathbb{R}_+^1 = (0, +\infty)$.

To construct u_1 we take the number t_1 such that $t_1 > 1$, $|G'(t_1)| \geq 1$ and t_1 is a Lebesgue point of G' . The latter implies that

$$(10) \quad \left| \{s : |t_1 - s| \leq \delta, |G'(s)| \geq 1\} \right| \geq \delta$$

for all sufficiently small δ . Next we choose $0 < r_1 < R$ so small that $t_1|B(0, r_1)|^{1/p} < 1/4$. For these t_1 and r_1 using (3) we find a ball $B(y_1, r_1)$, $y_1 \in \Omega$, and a function $v_1 \in V_\Omega(y_1)$ supported in $B(y_1, r_1) \cap \Omega$ such that $\int_\Omega |\nabla v_1|^p \leq 1/(4t_1)^p$, or

equivalently

$$\left(\int_{\Omega} |\nabla(2t_1v_1)|^p \right)^{1/p} \leq 1/2.$$

Now we put $w_1 = \min\{t_1, 2t_1v_1\}$. Note that truncating does not increase norm in $W^{1,p}(\Omega)$. Thus because of our choice of r_1 we have

$$\|w_1\|_{W^{1,p}(\Omega)} \leq \|2t_1v_1\|_{W^{1,p}(\Omega)} \leq 1.$$

Note that $v_1(y_1) = 1$. Consequently there exists a ball $B_1 = B(y_1, \rho_1) \subset\subset \Omega$ such that $w_1|_{B_1} \equiv t_1$. To finish construction of u_1 we use the function $\phi(x) = \max\{1 - |x|, 0\}$, $x \in \mathbb{R}^n$. We denote $\phi_\varepsilon(x) = \phi(x/\varepsilon)$ for $\varepsilon > 0$. We can choose $a > 0$ so small that (10) is satisfied for $\delta = a$, and $\varepsilon < \rho_1$ such that

$$\|a\phi_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq C_1(n)a\varepsilon^{n/p} \leq 1/2,$$

$$\int_{\mathbb{R}^n} |\nabla(a\phi_\varepsilon)|^p = C_2(n)a^p/\varepsilon^{p-n} = 1/2.$$

Note that (10) and the linear radial nature of ϕ imply that for such a

$$\left| \{z \in \mathbb{R}^n : |z| < \varepsilon, |G' \circ (t_1 + a\phi_\varepsilon(z))| \geq 1\} \right| \geq C_3(n)\varepsilon^n.$$

Therefore one also has

$$(11) \quad \int_{\mathbb{R}^n} |\nabla(G \circ (t_1 + a\phi_\varepsilon))|^p \geq C_3(n)a^p/\varepsilon^{p-n} \geq C_3(n)/2C_2(n).$$

Now taking $u_1(x) = w_1(x) + a\phi_\varepsilon(x - y_1)$, $x \in \Omega$, we see that $u_1 \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C_{loc}(\Omega)$ and (5), (7), (8) are valid for $j = 1$.

Now we construct u_k assuming that the functions u_1, \dots, u_{k-1} and the balls B_1, \dots, B_{k-1} with all required properties have already been constructed. We can find t_k such that

$$(12) \quad t_k \geq 4\|u_{k-1}\|_{L^\infty(\Omega)},$$

$|G'(t_k)| \geq k$, and t_k is a Lebesgue point of G' . Next we choose $0 < r_k < R$ so small that

$$(13) \quad 4r_k \leq \min\{\text{dist}(B_j, \partial\Omega), j = 1, \dots, k-1\},$$

and $t_k|B(0, r_k)|^{1/p} < 1/2^{k+2}$. For r_k, t_k we find, using (3), a ball $B(y_k, r_k)$, $y_k \in \Omega$, and a function $v_k \in V_\Omega(y_k)$ supported in $B(y_k, r_k) \cap \Omega$ such that

$$(14) \quad \left(\int_{\Omega} |\nabla(2t_kv_k)|^p \right)^{1/p} \leq 1/2^{k+2}.$$

We note that (9) and (14) imply that $\partial B(y_k, r_k) \cap \partial\Omega \neq \emptyset$. Thus $B(y_k, r_k)$ does not intersect any of B_1, \dots, B_{k-1} because of (13). We define w_k in a similar way as w_1 . Let $w_k = \min\{t_k, u_{k-1} + 2t_kv_k\}$. Because of (12) w_k coincides with u_{k-1} in the balls B_1, \dots, B_{k-1} . From (12) one has $u_{k-1}(y_k) + 2t_kv_k(y_k) > t_k$. Therefore there is a ball $B_k = B(y_k, \rho_k) \subset\subset \Omega$, $B_k \subset B(y_k, r_k)$, not intersecting B_1, \dots, B_{k-1} , such that $w_k|_{B_k} \equiv t_k$. By induction we have the estimate

$$\|w_k\|_{W^{1,p}(\Omega)} \leq \left(1 + \dots + \frac{1}{2^{k-1}} + \frac{1}{2^{k+1}} \right).$$

To construct u_k we modify w_k in the ball B_k . We use the function $a\phi_\varepsilon$ defined as above. We choose a and ε such that $\text{supp}(a\phi_\varepsilon) \subset B_k$,

$$(15) \quad \left| \{s : |t_k - s| \leq a, |G'(s)| \geq k\} \right| \geq a,$$

$$\|a\phi_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq 1/2^{k+1},$$

$$(16) \quad \int_{\mathbb{R}^n} |\nabla(a\phi_\varepsilon)|^p = 1/k^2.$$

By analogy with (11) from (15) and (16) we have

$$\int_{\mathbb{R}^n} |\nabla(G \circ (t_k + a\phi_\varepsilon))|^p \geq C_4(n)k^p/k^2 \geq C_5(n).$$

Now we define $u_k(x) = w_k(x) + a\phi_\varepsilon(x - y_k)$, and (5)–(8) hold. This completes the proof. \square

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