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A CO-FROBENIUS HOPF ALGEBRA WITH A SEPARABLE GALOIS EXTENSION IS FINITE

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ABSTRACT. If H is a co-Frobenius Hopf algebra over a field, having a Galois H-object A which is separable over A^{coH}, its ring of coinvariants, then H is finite dimensional.

Let H be a Hopf algebra over a field k and (A, ρ) a right H-comodule algebra such that A is H-Galois over its ring of coinvariants A^{coH}. If H is a group algebra kG so that then A is strongly graded, it is well known that if A is separable over A_e, its ring of coinvariants, then the group G is finite. We show in this note that this result holds for all co-Frobenius Hopf algebras H.

Recall that H is co-Frobenius if H^{*rat}, the rational submodule of H^* as a (left or right) H^*-module, is nonzero. If H^{*rat} is nonzero, it is dense in H^*. A right H-comodule algebra A is right H-Galois if the canonical map can: $A \otimes_{A^{coH}} A \to A \otimes H$, $a \otimes b \longmapsto \sum ab_0 \otimes b_1$, is a bijection. Any right H-comodule M is a left H^*-module via $p \cdot m = \sum m_0 \langle p, m_1 \rangle$ for $p \in H^*$, $m \in M$. Also H^* and H^{*rat} are right H-modules via $(p \leftharpoonup h)(f) = \langle p, hf \rangle$ for $p \in H^*$ or H^{*rat} and $h, f \in H$. Details about H-Galois objects with H co-Frobenius can be found in [1].

Lemma 1. *Suppose H is a Hopf algebra such that $H = \bigoplus_{\lambda \in I} H_\lambda$ where H_λ is a left H-subcomodule. Then if A is a right H-Galois object, $A = \bigoplus_{\lambda \in I} A_\lambda$ where $A_\lambda = \{a | a \in A,\ \rho(a) \in A \otimes H_\lambda\}$ and $A_\lambda \neq 0$ for all $\lambda \in I$.*

Proof. Let $p_\lambda \in H^*$ be the projection defined by $p_\lambda(h) = 0$ if $h \in H_\mu,\ \mu \neq \lambda$, and $p_\lambda(h) = \epsilon(h)$ for $h \in H_\lambda$. Then $p_\lambda \cdot H = \{\sum h_1 p_\lambda(h_2) | h \in H\} = H_\lambda$ since if $h \in H_\mu,\ \mu \neq \lambda,\ p_\lambda \cdot h = 0$, but if $h \in H_\lambda,\ p_\lambda \cdot h = \sum h_1 \epsilon(h_2) = h$.

Now, $p_\lambda \cdot A = \{\sum a_0 p_\lambda(a_1) | a \in A\}$ and so for $a \in A$,

$$\rho(p_\lambda \cdot a) = \sum a_0 \otimes a_1 p_\lambda(a_2) = \sum a_0 \otimes p_\lambda \cdot a_1 \in A \otimes H_\lambda$$

so that $p_\lambda \cdot A \subseteq A_\lambda$. Clearly if $a \in A_\lambda$, $p_\lambda \cdot a = a$ so $A_\lambda = p_\lambda \cdot A$, and $A = \bigoplus_{\lambda \in I} A_\lambda$.

Since A is Galois, the canonical map can from $A \otimes_{A^{coH}} A$ to $A \otimes H$, $a \otimes b \longmapsto \sum ab_0 \otimes b_1$ is a bijection and thus $A_\lambda \neq 0$ for all $\lambda \in I$. □

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If H is co-Frobenius, then H has a direct sum decomposition of left subcomodules

$$H = \bigoplus_{\lambda \in I} E(M_\lambda) = \bigoplus_{\lambda \in I} H_\lambda$$

where the M_λ's are the simple left subcomodules of H and the $E(M_\lambda)$'s their injective envelopes. By [4, Theorem 3], the $E(M_\lambda)$ are finite dimensional k-spaces. Thus H is finite dimensional if and only if I is finite.

Also since H is co-Frobenius, then H is semiperfect as a coalgebra, and the results of [2, Theorem 2.4] apply. In particular, for $H_\lambda = E(M_\lambda)$, the maps $p_\lambda \in H^*$ defined in Lemma 1 lie in H^{*rat} and the family of finite sums of the p_λ is a set of local units for H^{*rat}.

Theorem 2. *If H is co-Frobenius and A is H-Galois and separable over A^{coH}, then H is finite dimensional.*

Proof. Suppose that I is infinite. Let $\sum a_i \otimes b_i \in A \otimes_{A^{coH}} A$ be the separability idempotent for the extension A/A^{coH}; then

$$\sum a_i b_i = 1 \quad \text{and} \quad \sum c a_i \otimes b_i = \sum a_i \otimes b_i c \ \text{ for all } c \in A.$$

Since the maps p_λ can be used to build a set of local units $\sum_{j=1}^{n} p_{\lambda_j}$ for H^{*rat}, then if $f \in H^{*rat}$,

$$f = \sum_{\lambda \in F} f p_\lambda$$

for some finite subset F of I.

Let W be the finite dimensional subspace of H generated by the b_{i_1} and $p_W \in H^{*rat}$ a map equal to ϵ on W. The maps $p_W \leftharpoonup b_{i_1}$ are elements of H^{*rat}, and thus there is associated to each a finite subset F of I as above. Let λ_0 be an element of I which does not lie in any of these finite sets. In other words, $p_W \leftharpoonup b_{i_1}$ is zero on H_{λ_0} for all b_i.

Let $0 \neq c \in A_{\lambda_0}$ and applying $Id \otimes \rho$ to $\sum c a_i \otimes b_i = \sum a_i \otimes b_i c$, we obtain

$$\sum c a_i \otimes b_{i_0} \otimes b_{i_1} = \sum a_i \otimes b_{i_0} c_0 \otimes b_{i_1} c_1.$$

Now apply $M \cdot (Id \otimes Id \otimes p_W)$, M the multiplication in H, to both sides of the above equality and obtain

$$c = \sum c a_i b_{i_0} p_W(b_{i_1}) = \sum a_i b_{i_0} c_0 (p_W \leftharpoonup b_{i_1})(c_1) = 0,$$

which is a contradiction. Thus I is finite, and H is finite dimensional. □

Corollary 3. *If A/A^{coH} is separable and H-Galois, H co-Frobenius, then the equivalent conditions of* [3, *Proposition* 1.8] *hold.*

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