PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 11, Pages 3421–3423
S 0002-9939(00)05438-1

ON THE SET OF POINTS WITH A DENSE ORBIT

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(Communicated by Alan Dow)

Abstract. Under certain conditions on the topological space X we prove that for every continuous map $f : X \to X$ the set of all points with a dense orbit has empty interior in X. This result implies a negative answer to two problems proposed by M. Barge and J. Kennedy.

1. Introduction

Let X be a topological space. Given a subset A of X we shall denote by $\mathrm{Cl}_X(A)$ and $\mathrm{Int}_X(A)$ the closure and the interior of A in X, respectively. If U, V are subsets of $X \times X$ and $a \in X$, we define

$$U \circ V = \{(a,b) \in X \times X; (a,c) \in U \text{ and } (c,b) \in V \text{ for some } c \in X\}$$

and

$$U(a) = \{b \in X; (a,b) \in U\}.$$

Moreover, if $f : X \to X$ is a continuous map, the orbit of a point $x \in X$ under f is the set

$$O_f(x) = \{f^j(x); j \geq 0\}.$$

Finally, we shall denote by D_f the set of all points $x \in X$ whose orbit $O_f(x)$ is dense in X.

M. Barge and J. Kennedy [1] proposed the following problems (cf. Problems 5 and 6 on page 641 of [1]):

- Let $\{p_1, p_2, \dots, p_n\}$ be a set of $n \geq 2$ distinct points in the sphere S^2. Is there a homeomorphism of $S^2 - \{p_1, p_2, \dots, p_n\}$ such that every orbit of the homeomorphism is dense ?
- Is there a homeomorphism of $\mathbf{R}^n$, $n \geq 3$, such that every orbit of the homeomorphism is dense ?

Our goal is to answer both questions in the negative. More precisely, we shall see that if $X = S^2 - \{p_1, p_2, \dots, p_n\}$ or $X = \mathbf{R}^n$ $(n \geq 1)$, then any *continuous map* $f : X \to X$ has the property that $\mathrm{Int}_X(D_f) = \emptyset$ (in particular, $D_f \neq X$). In fact, our results apply to a much larger class of spaces X (cf. Theorem 1 and Corollary 6).

Received by the editors October 19, 1998 and, in revised form, January 22, 1999.
2000 *Mathematics Subject Classification.* Primary 37B20, 54H20.

2. Main results

Theorem 1. *Let X be a locally compact Hausdorff space which is not compact and has no isolated points. Then, for any continuous map $f : X \to X$, we have* $\mathrm{Int}_X(D_f) = \emptyset$.

Proof. Let Y be the one-point compactification of X and let $\mathcal{U}$ be the set of all open symmetric vicinities of the unique uniform structure compatible with the topology of Y. Fix $z \in \mathrm{Cl}_Y(X) - X$ and suppose that $\mathrm{Int}_X(D_f) \neq \emptyset$. Then, there are $x_0 \in X$ and $V \in \mathcal{U}$ such that

$$\mathrm{Cl}_Y(V(x_0)) \subset D_f.$$

For each $U \in \mathcal{U}$, choose an integer $r_U \geq 1$ such that $f^{r_U}(x_0) \in U(z)$. Let $\alpha_U \leq r_U$ be the greatest integer such that $f^{\alpha_U}(x_0) \in V(x_0)$ and let $\beta_U > r_U$ be the smallest integer such that $f^{\beta_U}(x_0) \in V(x_0)$. Put $y_U = f^{\alpha_U}(x_0)$. Then we have the following properties:

$$y_U, f^{\beta_U - \alpha_U}(y_U) \in V(x_0),$$

$$f(y_U), f^2(y_U), \dots, f^{\beta_U - \alpha_U - 1}(y_U) \notin V(x_0)$$

and

$$f^{r_U - \alpha_U}(y_U) \in U(z).$$

Let $y_0 \in Y$ be a cluster point of the net $(y_U)_{U \in \mathcal{U}}$. Since $y_0 \in \mathrm{Cl}_Y(V(x_0)) \subset D_f$, there is a smallest integer $M \geq 1$ such that

$$f^M(y_0) \in V(x_0).$$

Let $Z \in \mathcal{U}$ and $V' \in \mathcal{U}$ be such that

$$(f^M(y_0), x_0) \in V' \quad \text{and} \quad Z \circ V' \subset V.$$

For each $U \in \mathcal{U}$, let N_U be a neighborhood of y_0 in Y such that

$$(f^n(y), f^n(y_0)) \in Z \cap U \quad \text{for } n = 0, \dots, M, \quad \text{whenever } y \in X \cap N_U.$$

Since y_0 is a cluster point of $(y_U)_{U \in \mathcal{U}}$, there is a $W_U \in \mathcal{U}$, $W_U \subset U$, such that

$$y_{W_U} \in N_U.$$

Thus, $f^M(y_{W_U}) \in V(x_0)$. Since $\beta_{W_U} - \alpha_{W_U} \leq M$, we obtain

$$(f^{r_{W_U} - \alpha_{W_U}}(y_0), z) \in U \circ U.$$

Therefore, $z \in \mathrm{Cl}_Y(\{y_0, f(y_0), \dots, f^M(y_0)\})$, and so

$$z = f^j(y_0) \quad \text{for some } 0 \leq j \leq M.$$

But this is a contradiction, since $z \notin X$.

As an immediate consequence of Theorem 1 we obtain the following negative answers to the two problems mentioned in the introduction:

Corollary 2. *For any integer $n \geq 1$ and any continuous map $f : \mathbf{R}^n \to \mathbf{R}^n$, we have* $\mathrm{Int}_{\mathbf{R}^n}(D_f) = \emptyset$.

Corollary 3. *Let $X = S^2 - \{p_1, \dots, p_n\}$, where $p_1, \dots, p_n$ $(n \geq 1)$ are distinct points of S^2. Then, for any continuous map $f : X \to X$, we have* $\mathrm{Int}_X(D_f) = \emptyset$.

Remark 4. The conclusion of Corollary 2 is not true if we consider infinite-dimensional Banach spaces X in place of $\mathbf{R}^n$. Indeed, in certain infinite-dimensional Banach spaces X, Read [2] proved the existence of a continuous map $f : X \to X$ such that $D_f = X - \{0\}$ (his map is even linear). This also shows that we cannot omit the local compactness hypothesis in Theorem 1.

3. Further results

The argument used in the proof of Theorem 1 can also be applied to establish the following:

Theorem 5. *Let X be a compact Hausdorff space without isolated points. Then, for any continuous map $f : X \to X$, either $D_f = X$ or $\mathrm{Int}_X(D_f) = \emptyset$.*

Proof. Suppose $D_f \neq X$ and $\mathrm{Int}_X(D_f) \neq \emptyset$. Fix an $a \in X - D_f$. By arguing as in the proof of Theorem 1 (with $z = a$), we conclude that for some $y_0 \in D_f$ and some $j \geq 0$,

$$f^j(y_0) = z = a.$$

But this is a contradiction, since $y_0 \in D_f$ and $a \notin D_f$.

Corollary 6. *Let X be a compact convex set in a Hausdorff locally convex space and assume that X is not a singleton. Then, for any continuous map $f : X \to X$, we have $\mathrm{Int}_X(D_f) = \emptyset$.*

Proof. By the Schauder-Tychonoff fixed point theorem, f has a fixed point in X. So, we cannot have $D_f = X$.

Remark 7. (a) The possibility $D_f = X$ can happen in Theorem 5. For instance, let X be the unit circle and let $f : X \to X$ be a rotation by an irrational number.

(b) We cannot omit the hypothesis that X has no isolated points in Theorem 5. For instance, consider $X = \{0, 1\}$ and define $f : X \to X$ by $f(0) = 0$ and $f(1) = 0$. Then $D_f = \{1\}$, which is not X and does not have empty interior.

(c) In view of Read's example [2], we cannot omit the compactness hypothesis in Theorem 5.

References

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