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A CLASSIFICATION OF PRIME SEGMENTS IN SIMPLE ARTINIAN RINGS

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Abstract. Let A be a simple artinian ring. A valuation ring of A is a Bézout order R of A so that $R/J(R)$ is simple artinian, a Goldie prime is a prime ideal P of R so that R/P is Goldie, and a prime segment of A is a pair of neighbouring Goldie primes of R. A prime segment $P_1 \supset P_2$ is archimedean if $K(P_1) = \{a \in P_1 | P_1 a P_1 \subset P_1\}$ is equal to P_1, it is simple if $K(P_1) = P_2$ and it is exceptional if $P_1 \supset K(P_1) \supset P_2$. In this last case, $K(P_1)$ is a prime ideal of R so that $R/K(P_1)$ is not Goldie. Using the group of divisorial ideals, these results are applied to classify rank one valuation rings according to the structure of their ideal lattices. The exceptional case splits further into infinitely many cases depending on the minimal n so that $K(P_1)^n$ is not divisorial for $n \geq 2$.

1. Introduction

Dubrovin introduced in [D84] a class of valuation rings R, that are defined as Bézout orders in a simple artinian ring A so that $R/J(R)$ is again simple artinian. All one-sided finitely generated ideals of R are therefore principal and every element q in A can be written in the form $q = r_1 s_1^{-1} = s_2^{-1} r_2$ for r_i, s_i in R with s_i regular in R, $i = 1, 2$ (see [R67]). A rich extension theory in the finite dimensional case (for example see [D85], [BG90], [G92b], [MW89], and [W89]) suggests that this is the correct class of valuation rings in simple artinian rings.

The ideals of a valuation ring R in A are linearly ordered by inclusion and the overrings T of R in A are again valuation rings of A that are in one-to-one correspondence with the prime ideals P of R for which R/P is Goldie. We will call such prime ideals Goldie primes of R. If T is an overring of R, then the Jacobson radical $J(T)$ is a Goldie prime of R, and conversely if P is a Goldie prime of R, then $\mathcal{C}_R(P) = \{r \in R | r + P \text{ regular in } R/P\}$ is a *regular* Ore set in R and $T = R\mathcal{C}_R(P)^{-1} = R_P = {}_P R$ is an overring of R (see [G92a] or [MMU97], §14, for the general localization problem see [GW89], §12).

Let F be a skew field. A total valuation ring of F is a subring B of F so that $x \in F \backslash B$ implies $x^{-1} \in B$. Total valuation rings of F are exactly the Bézout orders B of F, for which $B/J(B)$ is a skew field.

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A prime ideal P in a total valuation ring is Goldie if and only if P is completely prime. The existence of non-Goldie primes in total valuation rings was raised as a problem in [BT76] and Dubrovin in [D93] constructed the first examples of such primes using the rational closure of group rings of certain left ordered groups and the universal covering group of $SL(2, \mathbb{R})$.

In the paper [BT76] it was also proved that a total valuation ring B of rank one, i.e. with $J(B)$ and (0) as its only completely prime ideals, is either invariant, i.e. $aR = Ra$ for all a in R, or has no other ideals besides $B, J(B)$ and (0), or contains a non-Goldie prime. In the present paper we show in Theorem 6 that an analogous classification holds for prime segments of valuation rings R in simple artinian rings A. Only the archimedean prime segments occur if A is finite dimensional over its center.

Every non-Goldie prime Q of R determines a prime segment $P_1 \supset P_2$ of R, i.e. $P_1 \neq P_2$ are Goldie primes and no further Goldie prime exists between P_1 and P_2 so that $P_1 \supset Q \supset P_2$, there are no further ideals between P_1 and Q, and $\bigcap Q^n = P_2$.

Essential for the proof is the result in Theorem 5 that $\bigcap I^n = P$ is a Goldie prime for every ideal $I \neq R$.

In the final section (see Theorem 9) the ideals of a rank one valuation ring R are completely described using the fact that such a ring is a maximal order and that the divisorial R-ideals of A form a group.

Prime segments can also be defined for cones of ordered or right ordered groups. They correspond to jumps (see [F66], [DD96], [BT97]).

2. Prime segments

Throughout this section we assume that A is a simple artinian ring and that R is a valuation ring of A; i.e. R is a Bézout order of A with $R/J(R)$ simple artinian for $J(R)$, the Jacobson radical of R. We will use various properties of such valuation rings that can be found in [D84] or [MMU97] where these rings are called Dubrovin valuation rings.

A prime ideal P of R is called a Goldie prime if R/P is Goldie, and the set of Goldie primes of R and the overrings of R are in one-to-one correspondence given by localization (see [G92a] and the introduction). A prime segment of R (and A) consists of two distinct Goldie primes $P_1 \supset P_2$ in R so that no further Goldie prime exists between P_1 and P_2. The ordinal type of the totally ordered set of prime segments of R is called the rank of R.

Proposition 1. *Let P_i be Goldie primes in R, $i \in \Lambda$. Then $P = \bigcap P_i$ is a Goldie prime.*

Proof. Since P_i is Goldie, the localization R_{P_i} exists for every i and we set $S = \bigcup R_{P_i}$ which is again a valuation ring of A and $J(S) \subseteq P_i$ since $R_{P_i} \subseteq S$. It follows that $J(S)$ is a Goldie prime contained in P ([G92a], [MMU97], §§6,14).

If we assume $P \supset J(S)$, then $PS = S$ since $R/J(S)$ is prime Goldie and $P/J(S)$ as a non-zero ideal in $R/J(S)$ contains a regular element $r + J(S)$ and r is regular in R ([G92a], Thm. 2.5) and a unit in $S = R_{J(S)}$.

We have $1 = \sum p_i s_i$ for elements p_i in P and s_i in S and there exists an index $j_0 \in \Lambda$ with $s_i \in R_{P_{j_0}}$ for all i.

Therefore, $1 = \sum p_i s_i \in P_{j_0} R_{P_{j_0}} = P_{j_0}$, a contradiction that shows $P = J(S)$ which is Goldie prime. $\square$

If I is an ideal in R, then $O_r(I) = \{q \in A | Iq \subseteq I\} = S$ is an overring of R, called the right order of I; the left order $O_\ell(I)$ is defined similarly.

Lemma 2. *Let $I \neq R$ be a non-zero ideal in R with $I = I^2$. Then I is neither a principal right $O_r(I)$-ideal nor a principal left $O_\ell(I)$-ideal.*

Proof. Both $S = O_r(I)$ and $T = O_\ell(I)$ are overrings of R and hence either $T \subseteq S$ or $S \subseteq T$, since the overrings of R are linearly ordered by inclusion. It is enough to consider the case $T \subseteq S$. We show first that $I \subseteq J(S)$. Otherwise, $I \supset J(S)$ and with the argument used in the proof of Proposition 1 it follows that $S = R_{J(S)} = IR_{J(S)} = IS = I$, a contradiction that proves $I \subseteq J(S)$. If I is principal as an S-right ideal, then $I = aS$ and a is a regular element. Therefore, $aS = I = I^2 = aSaS$ implies $S = SaS$ and $1 = \sum_{i=1}^n s_i a t_i$, s_i, $t_i \in S$, follows. Since $S = R_{J(S)} = {}_{J(S)}R$, there exist elements c, d in $\mathcal{C}_R\big(J(S)\big)$ with $cs_i \in R$, $t_i d \in R$ for all i. Hence $cd = \sum cs_i a t_i d \in I \subseteq J(S)$, a contradiction that proves that I is not a principal right S-ideal. Since $T \subseteq S$ implies $I \subseteq J(S) \subseteq J(T)$, a similar argument shows that I is not a principal left T-ideal. □

The next result shows that idempotent ideals $\neq R$ are Goldie primes.

Proposition 3. *Let $I^2 = I \neq R$ be an idempotent ideal in the valuation ring R. Then*

a) *$O_r(I) = S = O_\ell(I)$; and*
b) *$I = J(S)$ is a Goldie prime with $S = R_{J(S)}$.*

Proof. Let $S = O_r(I)$ and $T = O_\ell(I)$. It is enough to consider the case $S \subseteq T$. From Lemma 2 it follows that I is neither a principal right S-ideal nor a principal left T-ideal. Hence, $I^{-1}I = J(S)$ and $II^{-1} = J(T)$ for $I^{-1} = \{x \in A | IxI \subseteq I\}$ by [MMU97], 6.13(3). Further, $I^{-1} = \{x \in A | xI \subseteq S\} = (S : I)_\ell \supseteq T$. Conversely, if $x \in (S : I)_\ell$, then $xI \subseteq S$ and $xI = xI^2 \subseteq SI \subseteq I$, and $x \in T$ follows; we proved that $I^{-1} = T$. However, $II^{-1} = J(T) \subseteq R$ and $I^2 = I$ implies $T = I^{-1} \subseteq O_r(I) = S$ and $T = S$ follows which proves a).

We have $J(S) = I^{-1}I = TI = I$ which proves that I is Goldie since $J(S)$ is a Goldie prime; in addition, $S = R_{J(S)}$ follows and all statements in b) are proven. □

The next result shows that the union of Goldie primes is again a Goldie prime.

Corollary 4. *Let R be a valuation ring and let $R \supset P_i$, $i \in \Lambda$, be Goldie primes in R. Then:*

a) *$P = \bigcup P_i$ is Goldie prime;*
b) *$R_P = \bigcap R_{P_i}$;*
c) *$O_\ell(P) = R_P = O_r(P)$.*

Proof. If there exists a P_j with $P_j \supseteq P_i$ for all i, then $P = P_j$ is a Goldie prime, $R_P = R_{P_j} = \bigcap R_{P_i}$, and $P = J(R_P)$ implies $O_\ell(P) = O_r(P) = R_P$ ([MMU97], 6.8).

We can therefore assume that for every P_i there exists a P_j with $P_j \supset P_i$. Hence, $P \supset P_i$ for all i, and $P \supseteq P^2 \supset P_i$ for all i. It follows that $P = P^2$ is a Goldie prime with $R_P = O_\ell(P) = O_r(P)$. It remains to prove that $S = O_\ell(P)$ where $S = \bigcap R_{P_i}$. Let $x \in O_\ell(P)$, hence $xP \subseteq P$. Since $P \supset P_i$ is an ideal in R and R/P_i is Goldie, P contains an element in $\mathcal{C}(P_i)$ and $PR_{P_i} = R_{P_i}$. Therefore, $xR_{P_i} = xPR_{P_i} \subseteq PR_{P_i} = R_{P_i}$, and $x \in R_{P_i}$ for all i, $x \in S$ follows. Conversely, if

$x \in S$ and $a \in P$, then there exists P_j with $a \in P_j$ and $xa \in SP_j \subseteq R_{P_j}P_j = P_j \subset P$ proves $x \in O_\ell(P)$ and $S = O_\ell(P)$ follows. $\square$

Let $I \neq R$ be an ideal of R that is not a Goldie prime. Then it follows from Proposition 1 and Corollary 4 that there exists a prime segment $P_1 \supset P_2$ of R with $P_1 \supset I \supset P_2$.

Theorem 5. *Let $I \neq R$ be an ideal in the valuation ring R. Then $\bigcap I^n = P$ is Goldie prime.*

Proof. The result follows if $\bigcap I^n = I^m$ for a certain m, since then $(I^m)^2 = I^m$ is idempotent and we can apply Proposition 3. We can assume that $I^n \supset I^{n+1}$ and show that the assumption P not Goldie prime leads to a contradiction.

If I itself is a Goldie prime that does not have a lower neighbour among Goldie primes, then $I = \bigcup P_i$ for Goldie primes $I \supset P_i$. In this case, $I \supseteq I^2 \supset P_i$ for all i and hence $I = I^2$.

We can therefore assume that there exists a prime segment $P_1 \supset P_2$ in R with $P_1 \supseteq I \supset P_2$. We set $N = P_1 I P_1 \subseteq I$ and $I^3 \subseteq N$ and therefore $\bigcap I^n = \bigcap N^n = P$ follows; in addition, N and P are R_{P_1}-ideals. After localizing at P_1 we obtain $R_{P_1} \supset P_1 \supseteq N \supset \cap N^n = P \supset P_2$ and P is not Goldie prime in R_{P_1}. We therefore can consider R_{P_1}/P_2 and can assume from now on that R has rank one with $R \supset J(R) = P_1 \supseteq N \supset \bigcap N^n = P \supset (0)$.

We consider the following set W of ideals in $R:$

$$W = \{L | P_1 \supset L \supset P,\ L \text{ ideal of } R\}$$

and W contains N^n for $n \geq 2$.

In the first case we assume that W contains an ideal L which is not divisorial, i.e. $L \neq L^*$ where $L^* = \bigcap cR$ with $cR \supseteq L$ by the definition on p. 31 in [MMU97]. Here we use the fact that R is of rank one and hence $O_r(L) = R$. It follows from Proposition 6.13(1) in [MMU97] that $L^* = L^{-1^{-1}}$.

By [MMU97], 6.13(4), we have $L^* = aR$ and $L = aJ(R) = aP_1$. It follows that a is regular but not a unit in R and $L^* = aR \subseteq P_1$. Since $O_\ell(L^*) = aRa^{-1} = R$, we have $L^* = aR = Ra$ and $(L^*)^n = a^nR = Ra^n$ for $n \geq 1$.

It follows that the set $\mathcal{C} = \{a^n | n = 1, 2, \dots\}$ is an Ore system in R, the localization $R\mathcal{C}^{-1}$ contains R properly and $R\mathcal{C}^{-1} = A$ follows. Since P is a non-zero ideal in R, it contains a regular element c and $c^{-1} = ra^{-n}$ for some r in R and some $n \geq 1$. Hence, $a^n = cr \in P$, which implies $(L^*)^n = a^nR \subseteq P$, and the contradiction $L^* \subseteq P$, since otherwise $L^* \supseteq N^m$, for some m and $(L^*)^n \supseteq N^{mn} \supset P$.

In the second case we have $L = L^*$ for every ideal L in W and we consider L^{-1}. The R-ideal L^{-1} is divisorial and we claim that $L^{-1} \supset R$. Otherwise, $(L^{-1}L)^* = O_r(L) = R$, since R has rank one and the divisorial ideals form a group, but also $L^{-1} = R$ and the contradiction $(L^{-1}L)^* = L^* = L \subset R$ follows.

We consider

$$A_0 = \bigcup L^{-1}, \quad L \in W,$$

and want to prove that A_0 is an overring of R, hence equal to A. Let x, y be elements in A_0 and $x \in L_1^{-1},\ y \in L_2^{-1}$ for $L_1, L_2 \in W$ follows. Either $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$ and we can assume $L_1 \subseteq L_2$.

Since $L^{-1} = \{x \in A | LxL \subseteq L\} = \{x \in A | xL \subseteq O_r(L) = R\} = (R : L)_\ell$ for any non-zero ideal L of R, it follows that $L_2^{-1} \subseteq L_1^{-1}$ and $x \pm y \in L_1^{-1}$.

Further, $L_1^{-1}L_2^{-1}L_2L_1 \subseteq L_1^{-1}RL_1 \subseteq L_1^{-1}L_1 \subseteq R$ shows that $L_1^{-1}L_2^{-1} \subseteq (R : L_2L_1)_\ell = (L_2L_1)^{-1}$, and $xy \in (L_2L_1)^{-1}$. That $P_1 \supset L_1 \supset P$ and $P_1 \supset L_2 \supset P$ implies $P_1 \supset L_2L_1 \supset P$ follows since $P = \bigcap N^n$ and $N^{n+1} \subset N^n$ for all n.

To reach the final contradiction we choose a regular element c in $P \neq (0)$ and there exists L in W with $c^{-1} \in L^{-1}$. Therefore, $c^{-1}L \subseteq L^{-1}L \subseteq R$ and $L \subseteq cR \subseteq P$, but $P \subset L$. It follows that $P = \bigcap I^n$ is Goldie prime. □

Let Q be a prime ideal in R that is not Goldie. It follows from the remark before Theorem 5 that there exists a prime segment $P_1 \supset P_2$ in R with $P_1 \supset Q \supset P_2$. We call such a prime segment *exceptional.* Theorem 5 shows that $P_1 = P_1^2$, that there are no further ideals between P_1 and Q and that $\bigcap Q^n = P_2$.

On the other hand, we say that a prime segment $P_1 \supset P_2$ of R is *archimedean* if for every $a \in P_1 \backslash P_2$ there exists an ideal $I \subseteq P_1$ with $a \in I$ and $\bigcap I^n = P_2$. It follows from Theorem 5 that this will be exactly the case when either $P_1 \neq P_1^2$ or $P_1 = \bigcup I$, $I \subset P_1$, i.e. P_1 is the union of ideals I properly contained in P_1.

The next result shows that there are exactly three types of prime segments in valuation rings R.

Theorem 6. *For a prime segment $P_1 \supset P_2$ of a valuation ring R exactly one of the following possibilities occurs:*

a) *The prime segment $P_1 \supset P_2$ is archimedean;*

b) *The prime segment $P_1 \supset P_2$ is simple, i.e. there are no further ideals between P_1 and P_2;*

c) *The prime segment $P_1 \supset P_2$ is exceptional.*

Proof. We consider $L(P_1) = \bigcup I$, the union of ideals I of R properly contained in P_1. If $L(P_1) = P_2$, then the prime segment $P_1 \supset P_2$ is simple, characterizing the possibility b).

The prime segment $P_1 \supset P_2$ is exceptional if and only if $P_1 \supset L(P_1) \supset P_2$ and $P_1 = P_1^2$. If these conditions are satisfied and B and C are ideals of R properly containing $L(P_1)$, then $B \supseteq P_1$ and $C \supseteq P_1$ and $BC \supseteq P_1^2 = P_1$, which implies that $L(P_1)$ is prime but not Goldie. The converse was proved before stating the theorem.

We are left with the case that $P_1 \supset P_1^2$ and hence $\bigcap P_1^n = P_2$ or that $P_1 = \bigcup I$ for ideals I of R with $P_2 \subset I \subset P_1$. Again $\bigcap I^n = P_2$ for any such ideal by Theorem 5 and the prime segment $P_1 \supset P_2$ is archimedean. □

If we define $K(P_1) = \{a \in P_1 | P_1aP_1 \subset P_1\}$, it follows that $K(P_1)$ is an ideal in R and the following result holds:

Corollary 7. *The prime segment $P_1 \supset P_2$ of R is archimedean if and only if $K(P_1) = P_1$, it is simple if and only if $K(P_1) = P_2$, and it is exceptional if and only if $P_1 \supset K(P_1) \supset P_2$. In this last case, $K(P_1) = Q$ is a prime ideal that is not Goldie.*

It follows from this characterization that the type of the prime segment $P_1 \supset P_2$ is the same for any valuation ring R of A that contains this prime segment, in particular for R_{P_1}.

3. Rank one valuation rings

Let R be a rank one valuation ring of the simple artinian ring A; i.e. $J(R)$ and (0) are the only Goldie prime ideals of R. Since A is the only proper overring of

R, we have $O_\ell(I) = R = O_r(I)$ for every non-zero ideal I of R. The set $D(R)$ of divisorial R-ideals $(0) \neq I = I^*$ in A forms a group $D(R)$ with $(I_1 * I_2) = (I_1 I_2)^*$ as operation and R as identity ([MMU97], 6.15); we recall that $I^* = \bigcap cR$ for $cR \supseteq I,\ c \in A$.

This group contains the subgroup $H(R)$ of those non-zero ideals I which are principal right R-ideals, i.e. $I = aR$ for certain $0 \neq a \in A$. It follows that a is a unit in A and $O_\ell(aR) = aRa^{-1} = R$, hence $aR = Ra$. A non-zero R-ideal I in A is either divisorial or $I^* = aR = Ra$ and $I = aJ(R)$ and $J(R)$ is not a principal right R-ideal. The lattice of ideals of R is therefore known completely if $D(R)$ and $H(R)$ are known.

We observe that the intersection $K = \bigcap I_i$ of divisorial ideals I_i is again divisorial if K is not zero: $I_i \supseteq K$ implies $I_i^* = I_i \supseteq K^*$ and $K^* = K$ follows since $K^* \supseteq K$.

Lemma 8. *Let R be a rank one valuation ring. Then $D(R)$ is order isomorphic to a subgroup of $(\mathbb{R}, +)$, the additive group of real numbers.*

Proof. Let $I \subset R$ be a divisorial ideal. Then $\bigcap I^n = (0)$ by Theorem 5 if $I \subset J = J(R)$ or $I = J \neq J^2$, since R has rank one. If $J = J^2$, then J is not divisorial by [MMU97], 6.12. If $K = \bigcap (I^n)^* \neq (0)$, there exists therefore an integer k with $K \supset I^k$. By the remark made above it follows that K is divisorial; hence $K^* = K \supseteq (I^k)^*$ and $(I^k)^* \supset (I^{k+1})^*$ since $D(R)$ is a group. The contradiction $K \supset (I^{k+1})^*$ follows, the group $D(R)$ is archimedean and Hölder's Theorem ([F66], 74) shows that $D(R)$ is order isomorphic to a subgroup of $(\mathbb{R}, +)$. □

Theorem 9. *Let R be a rank one valuation ring of the simple artinian ring A with maximal ideal $J = J(R)$.*

Then exactly one of the following possibilities occurs:

a) *The segment $J \supset (0)$ is archimedean and*

  i) *$J \supset J^2$ and then $D(R) \cong \langle J \rangle \cong H(R)$ is an infinite cyclic group; or*

  ii) *$J = J^2$ and then $D(R) \cong (\mathbb{R}, +)$ and $H(R)$ is a dense subgroup of $D(R)$.*

b) *The segment $J \supset (0)$ is simple and then $D(R) = H(R) = \{R\}$ is the trivial group.*

c) *The segment $J \supset (0)$ is exceptional and Q with $J \supset Q \supset (0)$ is a non-Goldie prime in R. Then $D(R) = \langle Q \rangle$ is the infinite cyclic group generated by $Q = Q^*$ and an integer $k \geq 0$ exists with $H(R) = \langle (Q^k)^* \rangle$.*

Proof. We saw in Lemma 8 that $D(R)$ is an archimedean group. Assume that R contains a maximal divisorial ideal $I \subset R$, and let $C \subset R$ be any divisorial ideal. Then there exists a minimal n with $n \geq 1$ and $C \supseteq I^n$, hence $I^{n-1} \supset C \supseteq (I^n)^*$.

Therefore, $R = (I^{n-1})^* * (I^{-(n-1)})^* \supset C * (I^{-(n-1)})^* \supseteq I$ and, by the maximality of I, $I = C * (I^{-(n-1)})^*$ follows which implies $C = (I^n)^*$.

By [MMU97], 6.9, we have $J \neq J^2$ if and only if $J = aR = Ra$ and J is divisorial and a generator of the group $D(R)$; this proves the case a), i).

Next we consider the case a), ii) where $J = J^2$ and $J \supset (0)$ is an archimedean segment. For every non-zero element a in J exists therefore an ideal $I_1 \subset J$ with a in I_1 and hence $RaR \subset J$, using Theorem 6. We want to show that the ideal $I = RaR$ is a principal right R-ideal for any $0 \neq a$ in J and hence, $I \in H(R)$.

If I is not right principal, then $IJ = I$ ([MMU97], 6.9) and $a = \sum_{i=1}^n r_i a s_i$ for $r_i \in R,\ s_i \in J$ for all i. Since R is a left Bézout order, there exists s in R with $Rs_1 + \cdots + Rs_n = Rs$ and $s \in J$, $T_2 = RsR \subset J$ and $I = IT_2$ follows.

With $T_1 = J$ a regular ideal, $O_r(T_2) = R$ right Bézout we can apply the left-right symmetric version of 6.3 in [MMU97] to obtain a regular element t_0 in $T_1 = J$ so that $T_2 = RsR \subseteq Jt_0$. Hence, $I = IT_2 \subseteq IJt_0 \subseteq IJ = I$ and $I = IJt_0 = It_0$ follows for t_0 a regular element in J. Hence, t_0^{-1} exists in A and $It_0^{-1} = I,\ t_0^{-1} \in O_r(I) = R$, a contradiction since $t_0 \in J$.

This proves that $I = RaR$ is in $H(R)$ for any $0 \neq a \in J$. Finally, for every $RaR \subset J$ there exists $b \in J \backslash RaR$ and $RaR \subset RbR \subset J$ follows; $H(R)$ and $D(R)$ are therefore isomorphic to dense subgroups of $(\mathbb{R}, +)$. We observed before Lemma 8 that the intersection $K = \bigcap I_i$ of divisorial ideals of R is divisorial if $K \neq (0)$, hence $D(R)$ is also complete and $D(R) \cong (\mathbb{R}, +)$ follows.

It remains to consider the case c) where $J \supset Q \supset (0)$ is an exceptional prime segment. In this case, $J = J^2$ is not divisorial and $Q^* = Q$, since otherwise $Q^* \supset Q$, $J \supset Q$ and $Q = Q^*J$ leads to a contradiction for the prime ideal Q.

Therefore, Q is a maximal divisorial ideal in R, hence $D(R) = \langle Q \rangle$ by the first part of this proof and $H(R)$ is then equal to $\langle (Q^k)^* \rangle$ for some $k \geq 0$. □

We give lists of all ideals for rank one valuation rings in the case where $J(R) \supset (0)$ is exceptional.

If $k = 0$, then the proper ideals of R besides J and (0) are the powers of Q. If $k = 1$, then $Q^* = Q = aR = Ra$ is principal, $(Q^n)^* = a^nR = Ra^n$ and $R \supset J \supset aR \supset aJ \supset a^2R \supset a^2J \supset \cdots \supset (0)$ is the chain of ideals of R. If $k > 1$, then $(Q^k)^* = aR = Ra$ and $Q^k = aJ$. This follows, since the other possibility $Q^k = aR$ leads to $Q^kJ = aRJ = aJ$, hence $Q^k \supset Q^kJ$ and $Q \supset QJ$. By [MMU97], 6.9 we see that Q itself is a principal right R-ideal and the contradiction $k = 1$ follows.

In the case $k > 1$ we therefore have

$$\begin{aligned} R \supset J \supset Q \supset Q^2 \supset \cdots \supset Q^{k-1} \supset aR \supset aJ & \\ = Q^k \supset Q^{k+1} \supset \cdots \supset Q^{2k-1} \supset a^2R \supset a^2J & \\ = Q^{2k} \supset \cdots \supset (0). & \end{aligned}$$

We conclude with the discussion of some examples.

Example 10. Any discrete rank one commutative valuation domain R, like the rings of p-adic integers or the power series ring $K[[x]]$ over a field K, is an example to illustrate case a), i) in Theorem 9; the maximal ideal $J(R) = aR = Ra$ is principal and all other ideals $\neq (0)$ are powers of $J(R)$.

Let H be any dense subgroup of $(\mathbb{R}, +)$. Then one can construct (as Krull in 1932) a commutative valuation domain V as the localization of the subring KH^+ of the group ring KH for a field K and $H^+ = \{h \in H | h \geq 0\}$ at the multiplicatively closed set $S = \{\sum ha_h \in KH^+ | a_0 \neq 0\}$; i.e. $V = (KH^+)S^{-1}$ is a rank one valuation ring and the non-zero principal ideals of V have the form hV for $h \in H^+$. These rings, or $n \times n$ matrix rings over these rings, are examples for the case a), ii) in Theorem 9. Total rank one valuation rings R with $J \supset (0)$ archimedean are invariant, i.e. $aR = Ra$ holds for all a in R and if R is a rank one valuation ring in a simple artinian ring A finite dimensional over its center, then $J(R) \supset (0)$ is archimedean.

Example 11. To construct a total valuation ring R of rank one with a simple prime segment we consider (following Mathiak, see [M77]) the subgroup $H = \{\frac{m}{2^k} | m, k \in \mathbb{Z}\}$ of $(\mathbb{R}, +)$ and the valuation ring $V = (KH^+)S^{-1}$ constructed above. Then V

admits an automorphism σ defined by $\sigma(h) = 2h$ for h in H and we consider the Ore ring $V[x,\sigma] = \{\sum x^i a_i | a_i \in V\}$ with $ax = x\sigma(a)$ defining the multiplication. This ring contains the Ore system $T = \{\sum x^i a_i \in V[x,\sigma] \bigm|$ at least one a_i is a unit in $V\}$. Finally, $R = V[x,\sigma]T^{-1}$ is a rank one, total valuation ring with $J(R) \supset (0)$ simple: The non-zero principal right ideals of R have the form hR, $h \in H^+$ and $xhR = \frac{h}{2}R$. (See [BS95] for more examples.)

Example 12. To construct a rank one valuation ring with an exceptional prime segment we use results by Dubrovin ([D93], [DD96]).

The universal covering group G of $SL(2,\mathbb{R})$ contains a subsemigroup Π with $\Pi \cup \Pi^{-1} = G$ and $\Pi \cap \Pi^{-1} = U = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \bigm| b, 0 < a \in \mathbb{R}\}$ so that Π has rank one and contains a non-complete prime ideal. Here, a non-empty subset I of Π is a right ideal if $I\,\Pi \subseteq I$; ideals, prime ideals, completely prime ideals and the rank are defined similarly for Π.

Dubrovin shows that the group ring FG of G over a skew field F is embeddable into a skew field D that contains a rank one total valuation ring R with a prime ideal Q that is not completely prime, i.e. not Goldie. This construction can be modified in order to obtain examples for the various subcases in c) of Theorem 9.

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