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ASYMPTOTICALLY ISOMETRIC COPIES OF ℓ^∞ IN BANACH SPACES AND A THEOREM OF BESSAGA AND PEŁCZYŃSKI

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Abstract. We introduce the notion of a Banach space containing an asymptotically isometric copy of ℓ^∞. A well known result of Bessaga and Pełczyński states a Banach space X contains a complemented isomorphic copy of ℓ^1 if and only if X^* contains an isomorphic copy of c_0 if and only if X^* contains an isomorphic copy of ℓ^∞. We prove an asymptotically isometric analogue of this result.

1. Introduction

The notions of a Banach space containing an asymptotically isometric copy of c_0 or ℓ^1 have been introduced in [4], [5], [6]. The importance of these notions lie in the fact that if a Banach space contains an asymptotically isometric copy of c_0 or ℓ^1, then the Banach space fails the fixed point property for nonexpansive self mappings on closed bounded convex subsets of the Banach space [5], [6]. Examples are produced in [4] that show that the property of a Banach space containing an asymptotically isometric copy of ℓ^1 (respectively, an asymptotically isometric copy of c_0) is strictly stronger than the property of a Banach space containing an almost isometric copy of ℓ^1 (respectively, an almost isometric copy of c_0). It is proved in [4] that if a Banach space X contains an asymptotically isometric copy of c_0, then the dual space of X, X^*, contains an asymptotically isometric copy of ℓ^1. The converse of this result is easily seen to be false when one considers $X = \ell^1$, with its canonical norm. An isomorphic duality result which concerns Banach spaces containing c_0 is the following classical result of Bessaga and Pełczyński [1], [3], [11].

Theorem 1. *The following conditions are equivalent for a Banach space X:*

(1) *X^* contains an isomorphic copy of c_0,*
(2) *X contains a complemented isomorphic copy of ℓ^1,*
(3) *X^* contains an isomorphic copy of ℓ^∞.*

It is interesting to note that the isometric version of Theorem 1 does not hold. More precisely, the implications (1) implies (2), and (3) implies (2) do not hold if the word "isomorphic" is replaced by the word "isometric". This can be seen by

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considering $X = \ell^1(\Gamma)$, where Γ is an uncountable set. Let $\|\cdot\|_1$ (respectively, $\|\cdot\|_2$) denote the canonical norm on $\ell^1(\Gamma)$ (respectively, $\ell^2(\Gamma)$). Let $T : \ell^1(\Gamma) \to \ell^2(\Gamma)$ be the natural inclusion mapping. Now define an equivalent norm, $\|\cdot\|$, on $\ell^1(\Gamma)$ by

$$\|x\| = \left(\|x\|_1^2 + \|T(x)\|_2^2\right)^{\frac{1}{2}} \quad \text{for all} \quad x \in \ell^1(\Gamma).$$

Then $X = (\ell^1(\Gamma), \|\cdot\|)$ is a strictly convex Banach space and so X does not contain an isometric copy of ℓ^1. On the other hand, X^* is isomorphic to $\ell^\infty(\Gamma)$. Hence, since Γ is uncountable, X^* contains an isometric copy of ℓ^∞ (and so contains an isometric copy of c_0) [12].

The aim of this note is to prove an asymptotically isometric version of Theorem 1. We prove the following.

Theorem 2. *The following conditions are equivalent for a Banach space X:*

1. *X^* contains an asymptotically isometric copy of c_0,*
2. *there is a sequence $(x_n)_n$ in the unit ball of X and a bounded linear operator $S : X \to \ell^1$, with $\|S\| \le 1$ and $\lim\limits_{n\to\infty} \|Sx_n - e_n\| = 0$, where $(e_n)_n$ is the standard unit vector basis of ℓ^1,*
3. *X^* contains an asymptotically isometric copy of ℓ^∞.*

Remark. While conditions (1) and (3) of Theorem 2 are clearly asymptotically isometric analogues of conditions (1) and (3) of Theorem 1, condition (2) of Theorem 2 looks somewhat different than condition (2) of Theorem 1. However, in Theorem 2 (2) if we pass to subsequences we can assume that $\|Sx_n - e_n\| < 2^{-n}$ for all $n \in \mathbb{N}$. Then it is easy to see that the closed linear span of $(x_n)_n$ is asymptotically isometric to ℓ^1 and complemented in X. Moreover, for each $k \in \mathbb{N}$, there is a projection, P_k, from X onto the closed linear span of $(x_n)_{n\ge k}$ with $\lim\limits_{k\to\infty} \|P_k\| = 1$. This means that X contains asymptotically isometric copies of ℓ^1 which are complemented by projections of norm arbitrarily close to 1.

2. Definitions and preliminaries

In this section we will give the definitions of asymptotically isometric copies of c_0, ℓ^1 and ℓ^∞. The reader is referred to the texts of Diestel [3] and Lindenstrauss and Tzafriri [11] for any unexplained notation.

Definition 1. A Banach space X is said to contain an asymptotically isometric copy of ℓ^1 if there exists a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$\sum_n (1-\varepsilon_n)|t_n| \le \left\|\sum_n t_n x_n\right\| \le \sum_n |t_n| \ ,$$

for all $(t_n)_n \in \ell^1$.

The closed linear span of $(x_n)_n$ is called an asymptotically isometric copy of ℓ^1.

Definition 2. A Banach space X is said to contain an asymptotically isometric copy of c_0 if there exists a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$\sup_n (1-\varepsilon_n)|t_n| \le \left\|\sum_n t_n x_n\right\| \le \sup_n |t_n| \ ,$$

for all $(t_n)_n \in c_0$.

The closed linear span of $(x_n)_n$ is called an asymptotically isometric copy of c_0. If we wish to mimic Definitions 1 and 2 to obtain a definition of asymptotically isometric copies of ℓ^∞, we need to be more careful because if $(x_n)_n$ is an asymptotically isometric copy of ℓ^1 (respectively, c_0) and if $(t_n)_n \in \ell^1$ (respectively, $(t_n)_n \in c_0$), then $\sum\limits_{n=1}^{\infty} t_n x_n = \lim\limits_{k\to\infty} \sum\limits_{n=1}^{k} t_n x_k$. However, if $(t_n)_n \in \ell^\infty$ and $(x_n)_n$ is a sequence in X, then a formal series $\sum\limits_{n=1}^{\infty} t_n x_n$ will not necessarily be equal to $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{k} t_n x_n$ (in fact, while the formal series $\sum\limits_{n=1}^{\infty} t_n x_n$ may be well defined, it is possible that the partial sums $\left\{ \sum\limits_{n=1}^{k} t_n x_n \right\}_{k=1}^{\infty}$ might not converge). For this reason, our next definition will be given in operator theoretic terms.

Definition 3. A Banach space X is said to contain an asymptotically isometric copy of ℓ^∞ if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a bounded linear operator $T : \ell^\infty \to X$ so that

$$\sup_n (1-\varepsilon_n)|t_n| \le \|T((t_n)_n)\| \le \sup_n |t_n| \ ,$$

for all $(t_n)_n \in \ell^\infty$.

We will now identify alternative methods of recognizing asymptotically isometric copies of ℓ^1, c_0 and ℓ^∞

Proposition 1 ([8])**.** *A Banach space X contains an asymptotically isometric copy of ℓ^1 if and only if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\sum_{n=k}^{\infty} |t_n| \le \left\| \sum_{n=k}^{\infty} t_n x_n \right\| \le (1+\varepsilon_k) \sum_{n=k}^{\infty} |t_n|$$

for all $(t_n)_n \in \ell^1$ and for all $k \in \mathbb{N}$. Moreover, the closed linear span of $(x_n)_n$ is an asymptotically isometric copy of ℓ^1.

Proposition 2 ([7])**.** *A Banach space X contains an asymptotically isometric copy of c_0 if and only if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$(1-\varepsilon_k) \sup_{n\ge k} |t_n| \le \left\| \sum_{n=k}^{\infty} t_n x_n \right\| \le \sup_{n\ge k} |t_n| \ ,$$

for all $(t_n)_n \in c_0$ and for all $k \in \mathbb{N}$. Moreover, the closed linear span of $(x_n)_n$ is an asymptotically isometric copy of c_0.

By slightly modifying the proof of Proposition 2, we easily obtain the following:

Proposition 3. *A Banach space X contains an asymptotically isometric copy of ℓ^∞ if and only if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and an operator $T : \ell^\infty \to X$ so that*

$$(1-\varepsilon_k) \sup_{n\ge k} |t_n| \le \|T((t_n)_{n\ge k})\| \le \sup_{n\ge k} |t_n| \ ,$$

for all $(t_n)_n \in \ell^\infty$ and for all $k \in \mathbb{N}$.

3. The proof of Theorem 2

In Theorem 2, we note that (3) implies (1) trivially. We will separately consider the proofs of (1) implies (2), and (2) implies (3). We begin with the easier of the two implications; (2) implies (3).

Proof of (2) $\Rightarrow$ (3). Let $(\varepsilon_n)_n$ be a strictly decreasing null sequence in $(0,1/2)$. By passing to subsequences if necessary, we can assume that $\|Sx_n - e_n\| < \varepsilon_n$ for all $n \in \mathbb{N}$. Since $\|S\| \leq 1$, we have $\|S^*\| \leq 1$. Thus if $(a_n)_n \in \ell^\infty$ and $k \in \mathbb{N}$, then

$$\|S^*\left((a_n)_{n\geq k}\right)\| \leq \sup_{n\geq k} |a_n|.$$

On the other hand

$$\begin{aligned}
&\|S^*\left((a_n)_{n\geq k}\right)\| = \sup\left\{\left|\left[S^*\left((a_n)_{n\geq k}\right)\right](x)\right| : x \in X, \|x\| \leq 1\right\} \\
&= \sup\left\{\left|\left[(a_n)_{n\geq k}\right](Sx)\right| : x \in X, \|x\| \leq 1\right\} \\
&\geq \sup\left\{\left|\left[(a_n)_{n\geq k}\right]\left(\sum_{j=k}^\infty \beta_j S x_j\right)\right| : \sum_{j=k}^\infty \beta_j x_j \in X \text{ and } \left\|\sum_{j=k}^\infty \beta_j x_j\right\| \leq 1\right\} \\
&\geq \sup\left\{\left|\left[(a_n)_{n\geq k}\right]\left(\sum_{j=k}^\infty \beta_j S x_j\right)\right| : \sum_{j=k}^\infty |\beta_j| \leq 1\right\} \\
&\geq \sup\left\{\left|\left[(a_n)_{n\geq k}\right]\left(\sum_{j=k}^\infty \beta_j e_j\right)\right| - \left|\left[(a_n)_{n\geq k}\right]\left(\sum_{j=k}^\infty \beta_j (S x_j - e_j)\right)\right| : \sum_{j=k}^\infty |\beta_j| \leq 1\right\} \\
&\geq \sup\left\{\left|\sum_{j=k}^\infty a_n \beta_n\right| - \sup_{n\geq k}|a_n| \sum_{j=k}^\infty |\beta_j|\varepsilon_j : \sum_{j=k}^\infty |\beta_j| \leq 1\right\} \\
&\geq \sup_{n\geq k}|a_n| - \varepsilon_k \sup_{n\geq k}|a_n| \\
&= (1-\varepsilon_k)\sup_{n\geq k}|a_n|.
\end{aligned}$$

Thus we have shown that

$$(1-\varepsilon_k)\sup_{n\geq k}|a_n| \leq \|S^*\left((a_n)_{n\geq k}\right)\| \leq \sup_{n\geq k}|a_n| \ ,$$

for all $(a_n)_n \in \ell^\infty$ and for all $k \in \mathbb{N}$. Hence, by Proposition 3, X^* contains an asymptotically isometric copy of ℓ^∞. This completes the proof of (2) $\Rightarrow$ (3).

To complete the proof of Theorem 2 it remains only to prove (1) $\Rightarrow$ (2). For this we will need the following results.

Lemma 1 ([2]). *Let $(x_n)_n$ be a basic sequence in an infinite dimensional Banach space X. Then there is a block basic sequence $(y_n)_n$ of (x_n) and a sequence of functionals $(y_n^*)_n$ in X^* which form a unit biorthogonal system. That is, for each $n \in \mathbb{N}, \|y_n\| = \|y_n^*\| = y_n^*(y_n) = 1$ and $y_n^*(y_m) = 0$ for all $m \neq n$.*

Theorem 3 ([3], [10], [11]). *Let X be a separable infinite dimensional Banach space. If $(x_n^*)_n$ is a weak* null normalized sequence in X^*, then $(x_n^*)_n$ has a subsequence $(y_n^*)_n$ which is a weak* basic sequence.*

Theorem 4 ([9]). *Let X be a Banach space and let X_0 be a separable subspace of X. Then there exists a separable subspace Z of X which contains X_0, and an isometric embedding $J : Z^* \to X^*$ such that $(J(z^*))(z) = z^*(z)$ for all $z \in Z$ and $z^* \in Z^*$.*

Proof of (1) $\Rightarrow$ (2). We will first prove the implication when X is separable. We now assume that X is a separable Banach space and X^* contains an asymptotically isometric copy of c_0. Then there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence (x_n^*) in X^* so that

$$\sup_n (1-\varepsilon_n)|t_n| \leq \left\| \sum_n t_n x_n^* \right\| \leq \sup_n |t_n| \ ,$$

for all $(t_n)_n \in c_0$.

Since $(x_n^*)_n$ is a basic sequence in X^*, there is a block basic sequence $(y_n^*)_n$ of $(x_n^*)_n$ and a sequence of functionals (y_n^{**}) in X^{**} which form a unit biorthogonal system, by Lemma 1. Hence there is a strictly increasing sequence of integers, $(k_n)_{n=0}^\infty$ with $k_0 = 0$, and scalars, $\alpha_j^{(n)}$, where $k_{n-1}+1 \leq j \leq k_n$ and $n \in \mathbb{N}$, so that

$$y_n^* = \sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} x_j^* \ .$$

Since $\|y_n^*\| = 1$, we get

$$\max_{k_{n-1}+1 \leq j \leq k_n} (1-\varepsilon_j)|\alpha_j^{(n)}| \leq 1 \leq \max_{k_{n-1}+1 \leq j \leq k_n} |\alpha_j^{(n)}| \ .$$

In particular, $|\alpha_j^{(n)}| \leq (1-\varepsilon_j)^{-1} \leq [1 - \min_{k_{n-1}+1 \leq j \leq k_n} \varepsilon_j]^{-1}$ for all $k_{n-1}+1 \leq j \leq k_n$ and for all $n \in \mathbb{N}$. Define $\delta_n = 1 - \min_{k_{n-1}+1 \leq j \leq k_n} \varepsilon_j$, and let $z_n^* = \delta_n y_n^*$ for each $n \in \mathbb{N}$. Then for each $(a_n)_n \in c_0$ we have

$$\begin{aligned}
\left\| \sum_n a_n z_n^* \right\| &= \left\| \sum_n a_n \delta_n \left(\sum_{k_{n-1}+1 \leq j \leq k_n} \alpha_j^{(n)} x_j^* \right) \right\| \\
&= \left\| \sum_n \sum_{k_{n-1}+1 \leq j \leq k_n} a_n \delta_n \alpha_j^{(n)} x_j^* \right\| \\
&\leq \sup_{\substack{k_{n-1}+1 \leq j \leq k_n \\ n \in \mathbb{N}}} \left| a_n \delta_n \alpha_j^{(n)} \right| \leq \sup_{n \in \mathbb{N}} |a_n| \ .
\end{aligned}$$

Since $(x_n^*)_n$ is a sequence in X^* which is equivalent to the unit vector basis of c_0, and since $(y_n^*)_n$ is a block basis of $(x_n^*)_n, (y_n^*)_n$ is equivalent to the unit vector basis of c_0. In particular, $(y_n^*)_n$ is a weak* null sequence in X^*. Thus since X is separable and $(y_n^*)_n$ is a weak* null normalized sequence in X^*, $(y_n^*)_n$ has a subsequence (which we will again denote by $(y_n^*)_n$) which is weak* basic, by Theorem 3. By the construction of this sequence (see the proof of Theorem 3 [11, pages 11-12]), there is a bounded linear operator $T : X \to (\overline{\text{span}}\{y_n^*\}_{n=1}^\infty)^*$, defined by $(Tx)(y^*) = y^*(x)$, for all $y^* \in \overline{\text{span}}\{y_n^*\}_{n=1}^\infty$ and for all $x \in X$. Note that $\|T\| \leq 1$. Moreover, this operator has the property that for each $\varepsilon > 0$ and for each $y^{**} \in \text{span}\{y_n^{**}\}_{n=1}^\infty$ of norm 1, there exists an $x \in X$ with $\|x\| = 1$ and $\|Tx - y^{**}\| < \varepsilon$. Hence we have that for each $n \in \mathbb{N}$, there exists $x_n \in X$ with $\|x_n\| = 1$ so that $\|Tx_n - y_n^{**}\| < 2^{-n}$.

Define an operator $\Theta : (\overline{\text{span}}\{y_n^*\}_{n=1}^{\infty})^* \to \ell^1$, by $\Theta(y^{**}) = (y^{**}(z_n^*))_n$, for each $y^{**} \in (\overline{\text{span}}\{y_n^*\}_{n=1}^{\infty})^*$. Note that for each $n \in \mathbb{N}$, $\Theta(y_n^{**}) = \delta_n e_n$ and

$$\begin{aligned}
\|\Theta\| &= \sup\{|\Theta(y^{**})| : y^{**} \in (\overline{\text{span}}\{y_n^*\}_{n=1}^{\infty})^* \text{ and } \|y^{**}\| \leq 1\} \\
&= \sup\left\{\sum_{n=1}^{\infty} |y^{**}(z_n^*)| : y^{**} \in (\overline{\text{span}}\{y_n^*\}_{n=1}^{\infty})^* \text{ and } \|y^{**}\| \leq 1\right\} \\
&\leq \sup\left\{\|\sum_{n\in\Delta} \theta_n z_n^*\| : \Delta \text{ is a finite subset of } \mathbb{N} \text{ and } |\theta_n| = 1 \text{ for all } n \in \Delta\right\} \\
&\leq 1.
\end{aligned}$$

Finally, define $S : X \to \ell^1$ by $S = \Theta \circ T$. Then $\|S\| \leq \|\Theta\|\|T\| \leq 1$ and for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|Sx_n - e_n\| &= \|\Theta(Tx_n) - e_n\| \\
&= \|\Theta(Tx_n - y_n^{**}) + \Theta(y_n^{**}) - e_n\| \\
&\leq \|\Theta\|\|Tx_n - y_n^{**}\| + \|\Theta(y_n^{**}) - e_n\| \\
&< 2^{-n} + \|\delta_n e_n - e_n\| \\
&= 2^{-n} + 1 - \delta_n.
\end{aligned}$$

Hence $\lim_{n\to\infty} \|Sx_n - e_n\| = 0$ and so condition (2) is satisified.

For the general case, let X be a Banach space such that X^* contains an asymptotically isometric copy of c_0. Then there is a null sequence $(\varepsilon_n)_n$ in $(0, 1)$ and a sequence $(x_n^*)_n$ in X^* so that

$$\sup_n (1 - \varepsilon_n)|t_n| \leq \left\|\sum_n t_n x_n^*\right\| \leq \sup_n |t_n| \ ,$$

for all $(t_n)_n \in c_0$.

Let $Z = \overline{\text{span}}\{x_n^*\}_{n=1}^{\infty}$. Then Z is a separable subspace of X^*. Let $\{z_n\}_{n=1}^{\infty}$ be a countable dense subset of the unit ball of Z. For each $n \in \mathbb{N}$, choose a sequence $(x_{n,k})_k$ in the unit ball of X so that $\|z_n\| = \lim_{k\to\infty} z_n(x_{n,k})$. Now we define $Y = \overline{\text{span}}\{x_{n,k}\}_{n,k=1}^{\infty}$. Then Y is a separable subspace of X. By Theorem 4, there is a separable subspace Y_1 of X which contains Y and there is an isometric embedding $J : Y_1^* \to X^*$ satisfying $(Jy^*)(y) = y^*(y)$ for each $y^* \in Y_1^*$ and $y \in Y_1$.

It is clear from our construction that Z is isometric to the subspace $R(Z)$ of Y_1^*, where R is the restriction map from X^* to Y_1^*, and so Y_1^* contains an asymptotically isometric copy of c_0. Hence, since Y_1 is separable, the first part of the proof guarantees the existence of a sequence $(y_n)_n$ in the unit ball of Y_1 and an operator $S : Y_1 \to \ell^1$ with $\|S\| \leq 1$ and $\lim_{n\to\infty} \|Sy_n - e_n\| = 0$. By [3, page 114] there is a weakly unconditionally Cauchy series $\sum_n y_n^*$ in Y_1^* so that $S(y) = (y_n^*(y))_n$ for all $y \in Y_1$. Moreover,

$$\|S\| = \sup\left\{\|\sum_{n\in\Delta} \theta_n y_n^*\| : \Delta \text{ is a finite subset of } \mathbb{N} \text{ and } |\theta_n| = 1 \text{ for all } n \in \Delta\right\}.$$

For each $n \in \mathbb{N}$ define $x_n^* = J(y_n^*)$. Since J is an isometric embedding, the series $\sum_n x_n^*$ is a weakly unconditionally Cauchy series in X^* and the operator $S_0 : X \to$

ℓ^1 defined by $S_0(x) = (x_n^*(x))_n$ for all $x \in X$ satisfies $\|S_0\| = \|S\| \leq 1$. Also, for each $y \in Y_1$, $S_0(y) = (x_n^*(y))_n = ((J(y_n^*))(y))_n = (y_n^*(y))_n = S(y)$. Therefore, $\lim_{n\to\infty} \|S_0(y_n) - e_n\| = \lim_{n\to\infty} \|S(y_n) - e_n\| = 0$. This completes the proof.

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