

AN  $n$ -DIMENSIONAL SPACE THAT ADMITS  
A POINCARÉ INEQUALITY  
BUT HAS NO MANIFOLD POINTS

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ABSTRACT. For each integer  $n \geq 2$  we construct a compact, geodesic metric space  $X$  which has topological dimension  $n$ , is Ahlfors  $n$ -regular, satisfies the Poincaré inequality, possesses  $\mathbb{R}^n$  as a unique tangent cone at  $\mathcal{H}_n$  almost every point, but has no manifold points.

1. INTRODUCTION

In this paper, we shall construct, for each integer  $n \geq 2$ , a metric space  $X$  with the following properties:

- (i)  $X$  is compact, geodesic, and of topological dimension  $n$ ;
- (ii)  $X$  has Hausdorff dimension  $n$  and the Hausdorff  $n$ -measure  $\mathcal{H}_n$  satisfies the Ahlfors regularity condition:

$$(1.1) \quad \frac{1}{C} R^n \leq \mathcal{H}_n(B_R) \leq CR^n$$

for all balls  $B_R$  in  $X$  of radius  $R < \text{diam } X$  and for some  $C \geq 1$  independent of the ball;

- (iii)  $X$  admits the Poincaré inequality: there is a constant  $C \geq 1$  such that

$$(1.2) \quad \int_B |u - mv(u, B)| d\mathcal{H}_n \leq C \text{diam } B \int_B \rho d\mathcal{H}_n$$

for all integrable continuous functions  $u$  in a ball  $B$  and for all upper gradients  $\rho$  of  $u$  in  $B$ , where  $mv(u, B)$  denotes the mean-value of  $u$  in  $B$ ;

- (iv)  $X$  possesses  $\mathbb{R}^n$  as a unique tangent cone at  $\mathcal{H}_n$  almost every point;
- (v)  $X$  has no manifold points.

An additional property is that  $X$  is *uniformly rectifiable* in the sense of David and Semmes [DS]. A metric space is called *geodesic* if every pair of points in the space can be joined by a curve whose length is the distance between the points. A point in a metric space is a *manifold point* if it has a neighborhood that is homeomorphic

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to an open subset in Euclidean space. A *tangent cone* of a metric space  $(X, d)$  at a point  $p \in X$  is a pointed Gromov-Hausdorff limit of a sequence  $(X, \varepsilon_i^{-1}d, p)$  as  $\varepsilon_i \rightarrow 0$ ; cf. [GLP, 3.E]. An *upper gradient* of a function  $u$  in a metric space is any Borel function  $\rho$ , with values in  $[0, \infty]$ , such that

$$|u(x) - u(y)| \leq \int_{\gamma} \rho ds$$

for all pairs of points  $x, y$  in the space and for all rectifiable curves  $\gamma$  joining them. Observe that  $\rho = |\nabla u|$  is an upper gradient of a smooth function  $u$  on a Riemannian manifold.

Upper gradients and Poincaré inequalities in abstract metric spaces were introduced in [HeK] in connection with quasiconformal mappings. Recently, Cheeger [C] has exploited these concepts in studying differentiability of Lipschitz functions on metric spaces. (See [HaK 2] for further applications and references.) It was demonstrated in these papers that a lot of classical analysis, or first order calculus, can be accomplished in spaces that admit a Poincaré inequality, and a natural question therefore is: how pathological can a space be if (1.2) holds there?

Bourdon and Pajot [BP] showed that there exist metrics on the Sierpinski sponge such that (1.2) holds and that the Ahlfors regularity condition (1.1) holds for some nonintegral dimension. Shortly afterwards, Laakso [L] exhibited spaces that are Ahlfors regular of any given dimension greater than one, and where (1.2) holds. The spaces constructed by Bourdon and Pajot, and by Laakso, are all topologically 1-dimensional, and they have no manifold points. (Having heard of our construction, Laakso pointed out to us that his methods in [L] can be used to construct a space with properties (i) - (v). However, Laakso's spaces cannot be realized as compact subsets of some Euclidean space, up to a bi-Lipschitz change in the metric. It seems likely that our example has this property, but we have not checked the details.) The spaces that we construct in this paper have complicated local topology. One wonders if it is possible to have a space with properties (i) - (v) that is, say, locally contractible.

We thank Jeff Cheeger whose inquiries led us to think about the example presented in this paper.

## 2. GENERAL SET-UP AND THE MAIN PROPOSITION

All measures in this paper are locally finite Borel measures, nontrivial on each open set.

Our construction of a metric space  $X$  as in the Introduction is based on a proposition whose set-up is as follows:

$$(2.1) \quad \text{for each nonnegative integer } m, X_m = (X_m, d_m, \mu_m)$$

is a connected, compact metric measure space;

$$(2.2) \quad X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{\infty} = \bigcup_{m=0}^{\infty} X_m,$$

where the inclusion  $X_m \hookrightarrow X_{m+1}$  is an isometric embedding;

(2.3)  $X_m$ 's are uniformly doubling, that is, there exists a constant  $C \geq 1$ , independent of  $m$ , such that

$$\mu_m(B_m(x, 2r)) \leq C\mu_m(B_m(x, r)) \text{ whenever } x \in X_m \text{ and } r > 0,$$

where  $B_m(x, r) = \{y \in X_m : d_m(x, y) < r\}$ ;

(2.4)  $\mu_{m+1}|_{X_m} = \mu_m$ ;

(2.5) there is a real-valued function  $\varphi$ , independent of  $m$ , such that

$$\mu_m(B_m(x, r)) \leq \varphi(r) \text{ for all } x \in X_m \text{ and } r > 0.$$

We can equip  $X = X_\infty$  with a distance  $d = d_\infty$ , where

(2.6)  $d|_{X_m} = d_m$ ,

and with a Borel measure  $\mu = \mu_\infty$ , where

(2.7)  $\mu|_{X_m} = \mu_m$ .

It follows from (2.3) that the limiting space  $(X, d, \mu)$  is doubling as well:

(2.8)  $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ ,

where  $C \geq 1$  is as in (2.3) and  $B(x, r)$  is a ball in  $X$ .

If the center of a ball is irrelevant, we use the notation  $B(r)$ .

Next we describe two additional assumptions for this general set-up. For each  $m = 0, 1, 2, \dots$  let  $A_m = X_m \setminus X_{m-1}$ , where  $X_{-1} = \emptyset$ .

**Assumption A.** There exists a sequence  $D_0, D_1, \dots$  of measurable subsets of  $X$  with the following properties: Given any  $M > 0$  we have

(2.9)  $\mu(B(y, r)) \leq C(M)\mu(B^j(y, r))$

for all  $y \in D_j$  and  $r < MR_j(y)$ , where  $R_j(y)$  is the diameter of the  $y$ -component of  $D_j \cup D_{j+1}$  and  $B^j(y, r)$  is the  $y$ -component of  $B(y, r) \cap (D_j \cup D_{j+1})$ . Given  $a \in A_\ell$ ,  $y \in A_m$ ,  $m > \ell$ , with  $a$  and  $y$  in the same component of  $X_m \setminus X_{\ell-1}$ , there are points  $a = y_\ell, y_{\ell+1}, \dots, y_m = y$  with  $y_j \in D_j$  and positive numbers  $r_\ell, r_{\ell+1}, \dots, r_{m-1}$  such that

(2.10)  $r_j + r_{j+1} < CR_j = CR_j(y_j)$  for  $j = \ell, \dots, m - 2$

and  $y_j$  and  $y_{j+1}$  lie in the same component of  $D_j \cup D_{j+1}$  for  $j = \ell, \dots, m - 1$ . Moreover,

(2.11) 
$$\sum_{j=\ell}^{m-1} r_j \leq C d(a, y),$$

and

(2.12)  $d(y_j, y) \leq C r_j, \quad j = \ell, \dots, m - 1.$

Above,  $C \geq 1$  is a fixed constant (independent of  $a, y$ ).

**Assumption B.** Given  $x \in A_k$ ,  $y \in A_m$ , there exists  $\ell \leq \min\{k, m\}$  and  $a \in A_\ell$  such that

$$(2.13) \quad a \text{ and } x \text{ (resp. } a \text{ and } y) \text{ lie in the same component of } \\ X_k \setminus X_{\ell-1} \text{ (resp. } X_m \setminus X_{\ell-1}),$$

and that

$$(2.14) \quad d(x, a) + d(a, y) \leq C d(x, y),$$

where  $C \geq 1$  is a fixed constant (independent of  $x, y$ ). (Note that  $a$  may equal  $x$  or  $y$ .)

Before stating the main proposition, we define, for  $R > 0$ , the truncated maximal function  $M_R g$  of a (nonnegative) Borel function  $g$  in  $X$  by

$$M_R g(x) = \sup_{r < R} mv(g, B(x, r)),$$

where the notation  $mv(g, E)$  denotes the mean value of  $g$  over a measurable set  $E$ . In the proof of the proposition, we shall need maximal functions on subsets of  $X$  that are components of  $D_j \cup D_{j+1}$ ; these will be denoted by  $M_R^j$ , the component being understood from the context.

**Proposition 2.15.** *Let Assumptions A and B be satisfied. Suppose that  $u$  and  $\rho$  are measurable functions in  $B(R)$ , with  $\rho$  nonnegative, satisfying*

$$(2.16) \quad |u(x) - u(y)| \leq C_1 d(x, y) \left( M_{C_2 d(x, y)}^j \rho(x) + M_{C_2 d(x, y)}^j \rho(y) \right)$$

for all  $x, y \in B(R/C_3)$  that lie in the same component of  $D_j \cup D_{j+1}$  for some  $j$ . Then

$$(2.17) \quad |u(x) - u(y)| \leq C_4 d(x, y) (M_{C_5 R} \rho(x) + M_{C_5 R} \rho(y))$$

for all  $x, y \in B(R/C_6)$ , where the constants  $C_i$  for  $4 \leq i \leq 6$  depend only on the constants  $C_i$  for  $1 \leq i \leq 3$  and on the constants appearing in Assumptions A and B.

*Proof.* Fix  $x, y \in B(R/C_6)$ , where the constant  $C_6$  will be determined shortly. Assume that  $x \in A_k$  and  $y \in A_m$ . By Assumption B, we may choose  $a \in A_\ell$  for some  $\ell \leq \min\{k, m\}$  such that (2.13) and (2.14) are satisfied. It will be apparent from the proof that we may assume  $\ell < \min\{k, m\}$ . We use Assumption A and find  $y_\ell, \dots, y_m$  and  $r_\ell, \dots, r_{m-1}$  such that (2.10)-(2.12) hold. Therefore, we can fix  $C_6$  large enough so that

$$(2.18) \quad B^j = B^j(y_j, r_j) \subset B(R/C_3), \quad j = \ell, \dots, m-1.$$

For convenience, we also define  $B^m = \{y\}$  and  $r_m = 0$ . Let  $s_j = \min\{R_j, d(x, y)\}$ . We claim that, given  $\kappa > 0$ , there are points  $y'_j \in B^j$  such that

$$(2.19) \quad M_{\kappa s_j}^j \rho(y'_j) \leq N M_{CR} \rho(y), \quad j = \ell, \dots, m-1,$$

for some positive constants  $N$  and  $C$ , which depend only on the data and on  $\kappa$ . Below we will choose  $\kappa$  depending only on the data so that the same will be true for  $N$  and  $C$ . To prove the claim, fix  $j$  and let  $\kappa > 0$ . Then for  $r_j/2 \leq r < \kappa s_j$ , we have by (2.9), (2.8) and (2.12) that

$$(2.20) \quad mv(\rho, B^j(z, r)) \leq C mv(\rho, B(y, Cr)) \leq C M_{CR} \rho(y)$$

for every  $z \in B^j$  and for some  $C$  depending only on the data. Therefore, if (2.19) fails to hold for some large  $N$ , then for every  $z \in B^j$ ,  $d(y_j, z) < r_j/2$ , there exists  $r(z) < r_j/2$  such that

$$mv(\rho, B^j(z, r(z))) > NM_{CR} \rho(y).$$

Thus, by using standard covering arguments and the fact that the measure  $\mu$  is doubling on  $D_j \cup D_{j+1}$  by (2.9), we obtain that

$$(2.21) \quad mv(\rho, B^j) > CNM_{CR} \rho(y).$$

On the other hand, by (2.20),

$$mv(\rho, B^j) \leq C M_{CR} \rho(y),$$

which contradicts (2.21) for  $N$  large enough. Thus, (2.19) follows.

Next, note that from (2.10), (2.11) and (2.14) we get

$$r_j + r_{j+1} < Cs_j, \quad j = \ell, \dots, m - 2.$$

From this inequality and (2.12), it follows that we can (and do) choose  $\kappa$  such that

$$(2.22) \quad C_2 d(w_j, w_{j+1}) \leq \kappa s_j, \quad j = \ell, \dots, m - 1$$

for all  $w_j \in B^j$  and  $w_{j+1} \in B^{j+1}$ . Next, note that for  $j = \ell, \dots, m - 2$  we have

$$d(y'_j, y'_{j+1}) \leq C(r_j + r_{j+1})$$

by (2.12). Moreover, if we define  $y'_m = y$ , we have  $d(y'_{m-1}, y'_m) \leq Cr_{m-1}$  so that

$$(2.23) \quad \sum_{j=\ell}^{m-1} d(y'_j, y'_{j+1}) \leq C d(a, y)$$

by (2.11). It follows from Assumption A that  $y'_j, y'_{j+1}$  lie in the same component of  $D_j \cup D_{j+1}$  for  $j = \ell, \dots, m - 1$ . Therefore, by (2.18), (2.19), (2.22), and assumption (2.16)

$$(2.24) \quad \begin{aligned} |u(y'_j) - u(y'_{j+1})| &\leq C d(y'_j, y'_{j+1}) \left( M_{C_2 d(y'_j, y'_{j+1})}^j \rho(y'_j) + M_{C_2 d(y'_j, y'_{j+1})}^j \rho(y'_{j+1}) \right) \\ &\leq C d(y'_j, y'_{j+1}) M_{CR} \rho(y) \end{aligned}$$

for all  $j = \ell, \dots, m - 1$ . It hence follows from (2.23) that

$$(2.25) \quad |u(y'_\ell) - u(y)| \leq C d(a, y) M_{CR} \rho(y).$$

Similarly, we also have

$$(2.26) \quad |u(x'_\ell) - u(x)| \leq C d(a, x) M_{CR} \rho(x)$$

for some  $x'_\ell \in B^\ell$ . Finally, because  $y'_\ell$  and  $x'_\ell$  lie in the same component of  $D_\ell$  with  $d(x'_\ell, y'_\ell) \leq C d(x, y)$  by (2.11) and (2.14), we have by assumption (2.16) as in the proof of (2.19) that

$$(2.27) \quad |u(x'_\ell) - u(y'_\ell)| \leq C d(x, y) (M_{CR} \rho(x) + M_{CR} \rho(y)).$$

The proposition now follows by combining (2.14), and (2.25)-(2.27).

## 3. THE EXAMPLE

We now verify the claim made in the Introduction by constructing a metric space with properties (i) - (v). Before beginning our construction, we record the following two results.

**Theorem 3.1.** *A doubling metric measure space  $X$  admits a weak Poincaré inequality if and only if there is a constant  $C \geq 1$  such that*

$$(3.2) \quad |u(x) - u(y)| \leq Cd(x, y) \left( M_R \rho(x) + M_R \rho(y) \right)$$

whenever  $u$  is a continuous function in  $B(R)$ ,  $x, y \in B(R/C)$ , and  $\rho$  is an upper gradient of  $u$  in  $B$ .

**Theorem 3.3.** *Let  $X, (Y_i)$  be a countable collection of metric measure spaces that are uniformly Ahlfors  $n$ -regular (condition (1.1)). Suppose  $(F_i)$  is a disjoint collection of compact subsets of  $X$  such that each  $F_i$  has an isometric copy (also called  $F_i$ ) inside  $Y_i$ . Suppose further that  $n \geq s > n - 1$  and that*

$$(3.4) \quad \mathcal{H}_s^\infty(F_i \cap B(R)) \geq C^{-1} R^s$$

for all balls  $B(R)$  in  $X$  or in  $Y_i$  centered at  $F_i$ , where  $C \geq 1$  is independent of  $i$  and  $0 < R < \min\{\text{diam } X, \text{diam } Y_i\}$ . Then, if  $X$  and  $Y_i$  admit a weak Poincaré inequality with constants independent of  $i$ , then the space  $X \cup_{(F_i)} Y_i$  admits a weak Poincaré inequality as well. The statement is quantitative in the usual sense.

Recall that a space admits a *weak* Poincaré inequality if (1.2) holds with the ball  $B$  on the left hand side replaced by  $\lambda B$  for some fixed  $0 < \lambda \leq 1$ ; cf. [HeK] Section 5. In (3.4),  $\mathcal{H}_s^\infty$  denotes the Hausdorff  $s$ -content of a set.

The metric space  $X \cup_{(F_i)} Y_i$  is obtained by glueing each  $Y_i$  to  $X$  along  $F_i$ ; the metric is given by  $d(a, b) = \inf_{p \in F_i} (d_{Y_i}(a, p) + d_X(p, b))$ , if  $a \in Y_i$  and  $b \in X$ , and by  $d(a, b) = \inf_{p \in F_i, q \in F_j} (d_{Y_i}(a, p) + d_X(p, q) + d_{Y_j}(q, b))$ , if  $a \in Y_i$ ,  $b \in Y_j$ , and  $i \neq j$ .

The maximal function characterization of the Poincaré inequality, Theorem 3.1, was proved in [HeK, Lemma 5.15] (it was assumed there that  $X$  be locally compact, but this assumption is redundant). Theorem 3.3 was proved in [HeK, Theorem 6.15] for the case of two spaces  $X, Y$  (again, the local compactness assumption is redundant). The more general case described in Theorem 3.3 easily reduces to the special case by way of the maximal function characterization in Theorem 3.1. Note that by using Theorem 3.3 alone it is relatively easy to construct spaces with the Poincaré inequality and large nonmanifold pieces. To obtain a space with no manifold points at all, we require an iterative construction together with a repeated use of Theorem 3.1.

We now begin the required construction in earnest. Fix an integer  $n \geq 2$ . Next, fix a compact totally disconnected set  $E \subset [0, 1]$  such that

$$(3.5) \quad |E|_1 \geq \frac{2}{3},$$

and that

$$(3.6) \quad \mathcal{H}_{1/2}^\infty(E \cap B(r)) \geq C^{-1} r^{1/2}$$

for each ball (interval)  $B(r)$  centered at  $E$  with  $0 < r \leq 1$ . We leave it to the reader to construct such a set  $E$ . Here and below, we let  $|\cdot|_j$  denote Lebesgue  $j$ -measure in  $\mathbb{R}^j$ .

Now let  $I^j = [0, 1]^j$ , a closed  $j$ -cube in  $\mathbb{R}^j$ , and  $F = E \times I^{n-1}$ . Then  $F$  is a compact subset of  $I^n$  with empty interior satisfying

$$(3.7) \quad |F|_n \geq \frac{2}{3},$$

and

$$(3.8) \quad \mathcal{H}_{n-\frac{1}{2}}^\infty(F \cap B(r)) \geq C^{-1}r^{n-\frac{1}{2}}$$

for each  $n$ -ball  $B(r)$  centered at  $F$  with radius  $0 < r \leq \sqrt{n}$ . By shrinking  $F$  slightly, we may assume, in addition, that

$$(3.9) \quad \text{dist}(F, \partial I^n) = \delta > 0.$$

Here and below, a *cube* will refer to a closed dyadic sub- $n$ -cube of  $I^n$ . Given a cube,  $Q$ , we define  $F_Q = T_Q(F)$ , where  $T_Q : I^n \rightarrow Q$  is the canonical similarity map onto  $Q$ ; that is,  $T_Q(x) = \ell(Q)x + z_Q$ , where  $\ell(Q)$  is the side length of  $Q$  and  $z_Q$  is the vertex of  $Q$  which is closest to the origin. Then  $F_Q$  is a compact subset of  $Q$  with empty interior satisfying

$$(3.10) \quad |F_Q|_n \geq \frac{2}{3} |Q|_n,$$

$$(3.11) \quad \mathcal{H}_{n-\frac{1}{2}}^\infty(F_Q \cap B(r)) \geq C^{-1}r^{n-\frac{1}{2}}$$

for each  $n$ -ball  $B(r)$  centered at  $F_Q$  with radius  $0 < r < \text{diam } Q$ , and

$$(3.12) \quad \text{dist}(F_Q, \partial Q) = \delta \ell(Q).$$

Define  $X_0 = I^n$ ,  $\mu_0 =$  Lebesgue  $n$ -measure restricted to  $X_0$ , and  $d_0 =$  the standard Euclidean distance in  $\mathbb{R}^n$ . Let  $W_1 = \{Q_m\}$  be a Whitney decomposition of  $\mathbb{R}^n \setminus F$  restricted to  $X_0 \setminus F$ ; thus the cubes in  $W_1$  are dyadic subcubes of  $I^n$ . (Compare [S, p. 16].) Then for each  $Q$ ,  $W_1$  determines a Whitney decomposition  $W_Q$  of  $Q \setminus F_Q$ ; namely,  $W_Q = \{T_Q(Q_m) : Q_m \in W_1\}$ . Define  $W_2 = \bigcup_{Q \in W_1} W_Q$  and more generally for each integer  $k \geq 1$ ,  $W_{k+1} = \bigcup_{Q \in W_k} W_Q$ . Let  $W = \bigcup_{k=1}^\infty W_k$ , and set  $X_1 = X_0 \cup_{(F_Q)} Q'$ , where  $Q'$  is an identical copy of  $Q$ , and  $(F_Q)$  is indexed by  $Q \in W$ .

As explained above after Theorem 3.3, there is a natural extension of  $d_0$  to a metric  $d_1$ , which is geodesic in  $X_1$ . Similarly,  $\mu_0$  extends naturally to a measure  $\mu_1$ :

$$\mu_1(K) = \mu_0(K \cap X_0) + \sum_{Q \in W} \mu_0(K \cap (Q' \setminus F_Q))$$

for each Borel set  $K \subset X_1$ .

Thus,  $X_1$  is  $X_0$  together with a countable number of "flaps" glued on. Each flap is a cube  $Q'$  which is attached to  $X_0$  at the compact interiorless set  $F_Q$ , a scaled version of the initial set  $F$ . Now to get  $X_2$  from  $X_1$ , we glue flaps onto each flap in  $X_1$  in exactly the same way that we glued flaps onto  $X_0$  to create  $X_1$ . In other words, we replace each flap  $Q'$  by an appropriate scaled copy of  $X_1$ . Metric  $d_1$  and measure  $\mu_1$  extend to  $d_2$  and  $\mu_2$  in the canonical way. By repeating this process inductively, we create a sequence of compact and geodesic metric measure spaces

$X_m = (X_m, d_m, \mu_m)$ . We let  $X = \bigcup_{m=0}^\infty X_m$  as in Section 2; note that (2.1), (2.2), and (2.4) are satisfied. We define the metric  $d$  and measure  $\mu$  on  $X$  by formulas (2.6) and (2.7).

We shall show that  $X$  satisfies properties (i) - (v) in the Introduction except for compactness; in the end, a simple argument will show that the completion of  $X$  will satisfy all requirements. Observe that the size of the flaps goes down rapidly so that  $\text{diam } X \leq 2\sqrt{n}$ .

Properties (iv) and (v) are quite clear from the construction, for almost every point in  $X$  belongs to some  $F_Q$  or its iterative counterpart.

The next order of business is to show that (2.3) and (2.5) hold, so that all of the properties (2.1) - (2.8) in the basic set-up in Section 2 are satisfied. To that end, we actually show that

$$(3.13) \quad \frac{1}{C} R^n \leq \mu_m(B(R)) \leq CR^n$$

for each ball  $B(R) \subset X_m$ ,  $0 < R < \text{diam } X_m$ , and for some  $C \geq 1$  independent of  $m$  and  $R$ ; in other words, the spaces  $X_m$  are uniformly Ahlfors  $n$ -regular. This implies both (2.3) and (2.5). We shall show, in effect, that (3.13) holds with  $X$  in place of  $X_m$  and  $\mu$  in place of  $\mu_m$ , but a similar argument gives (3.13).

To prove (3.13), we set up some notation. First, recall that for  $k = 0, 1, 2, \dots$ ,  $A_k = X_k \setminus X_{k-1}$ , where  $X_{-1} = \emptyset$ . Given  $x \in A_k$ ,  $y \in A_\ell$ , with  $\ell \geq k$ , we say that  $y$  is a *descendant* of  $x$  if there is a path connecting  $x$  to  $y$  in  $X_\ell \setminus X_{k-1}$ . We let  $D(x)$  denote the set of all descendants of  $x \in X$ , and  $D_m(x) = D(x) \cap X_m$ .

Throughout the rest of this paper,  $C$  denotes a positive constant that only depends on some obvious data.

To verify the lower bound in (3.13), note that if  $x \in X$ , then  $x \in A_m$  for some  $m \geq 0$ . If  $m = 0$ , the lower bound is trivial. If  $m \geq 1$ , then there is a sequence of flaps  $Q'_1, \dots, Q'_m$  that “connect”  $X_0 = Q_0$  to  $x \in Q'_m$ . Because the diameter of these consecutive flaps decreases at least geometrically, we have that  $\text{dist}(Q'_j, x) \leq C \text{diam } Q'_j$  for each  $j = 0, \dots, m$ . Thus, if some  $Q'_j \subset B(x, R) \setminus B(x, R/2)$ , then  $\text{diam } Q'_j \geq R/C$ , which implies the required lower bound; if not, there must be a cube  $Q'_j$  that either connects  $\partial B(x, R/2)$  to  $\partial B(x, 3R/4)$ , or  $\partial B(x, 3R/4)$  to  $\partial B(x, R)$ , in which case the lower bound again holds.

Next, we shall verify the upper bound in (3.13). To that end, it suffices to show that

$$(3.14) \quad \mu(B(x, R) \cap D(x)) \leq CR^n$$

if  $x \in X$  and  $0 < R < \text{diam } X$ . (Indeed, if  $j$  is the smallest integer such that  $B(x, R) \cap X_j \neq \emptyset$ , then for  $y \in B(x, R) \cap X_j$  we have  $B(x, R) \subset B(y, 2R) \cap D(y)$ .) In order to prove (3.14), note first that for any cube  $Q$  we have  $\bigcup_{P \in W_Q} P = Q \setminus F_Q$ , and hence (3.10) implies that

$$\mu\left(\bigcup_{P \in W_Q} P\right) \leq \frac{1}{3} \mu(Q).$$

From this it follows that

$$\mu\left(\bigcup_{Q \in W_{m+1}} Q\right) \leq \frac{1}{3} \mu\left(\bigcup_{Q \in W_m} Q\right) \leq \frac{1}{3^{m+1}} \mu(X_0) = \frac{1}{3^{m+1}}$$

for  $m = 1, 2, \dots$ , and hence that  $\mu\left(\bigcup_{Q \in W} Q\right) \leq \frac{1}{2}$ . Therefore

$$\mu(X_1 \setminus X_0) \leq \frac{1}{3} \mu\left(\bigcup_{Q \in W} Q\right) \leq \frac{1}{6}.$$

From these estimates and from the construction of  $X_2$ , we obtain that

$$\mu(X_2 \setminus X_1) \leq \frac{1}{6} \mu\left(\bigcup_{Q \in W} Q\right) \leq \frac{1}{12},$$

and, in general, that  $\mu(X_m \setminus X_{m-1}) \leq \frac{1}{3} \left(\frac{1}{2}\right)^m$  for  $m = 1, 2, \dots$ . It follows that

$$(3.15) \quad \mu(X) \leq \frac{4}{3} = \frac{4}{3} \mu(X_0),$$

and by self-similarity of the construction that

$$(3.16) \quad \mu\left(D(Q')\right) \leq \frac{4}{3} \mu(Q') = \frac{4}{3} \mu(Q)$$

for each  $Q'$  which is a flap at some stage of the construction, where  $D(Q') = \bigcup_{x \in Q'} D(x)$ .

Finally, to prove (3.14), we may assume that  $x \in X_0$ , by self-similarity of the construction. Then  $D(x) = X$ , and we must show that

$$(3.17) \quad \mu\left(B(x, R)\right) \leq CR^n.$$

It is easy to see that there is at most one  $Q \in W$ , call it  $Q_0$ , such that both  $F_{Q_0} \cap B(x, R) \neq \emptyset$  and  $B(x, R) \cap X_0 \subset Q_0$ . Let

$$W^* = \left\{ Q \in W : Q \neq Q_0, \quad F_Q \cap B(x, R) \neq \emptyset \right\}.$$

Note that  $W^*$  is nonempty. In addition, it follows from (3.12), and from the fact that none of the  $Q$ 's in  $W^*$  contain  $B(x, R) \cap X_0$ , that

$$(3.18) \quad Q \subset B(x, CR)$$

for each  $Q \in W^*$ . Let  $W^{**}$  be the collection of all maximal  $Q$ 's from  $W^*$ ; that is,  $Q \in W^{**}$  if  $Q \subset P$  and  $P \in W^*$  imply  $Q = P$ . If we denote, for  $Q \in W^{**}$ ,  $E(Q) = \{P \in W^* : P \subset Q\}$ , we have that

$$\bigcup_{Q \in W^{**}} E(Q) = \bigcup_{Q \in W^*} Q,$$

and that  $E(Q) \cap E(P) = \emptyset$  if  $Q \neq P$ . Now for  $Q \in W^{**}$  the ratio of

$$\mu\left(\left(\bigcup_{P \in E(Q), P \neq Q} D(P') \setminus Q\right)\right)$$

and  $\mu(Q)$  is less than or equal to the ratio of  $\mu(X \setminus X_0)$  and  $\mu(X_0)$ , which does not exceed  $\frac{1}{3}$  by (3.15). Also, since  $\mu(D(Q') \setminus Q) \leq \frac{2}{3} \mu(Q)$  by (3.10) and (3.16), we have that

$$(3.19) \quad \mu\left(\left(\bigcup_{P \in E(Q)} D(P') \setminus Q\right)\right) \leq \mu(Q).$$

Next, abbreviate  $B = B(x, R)$ . We note that

$$B = (B \cap X_0) \cup \left( B \cap \left( D(Q'_0) \setminus Q_0 \right) \right) \cup \left( B \cap \left( \bigcup_{Q \in W^*} D(Q') \setminus Q \right) \right).$$

But the last of the three sets in this union belongs to

$$\bigcup_{Q \in W^*} D(Q') \setminus Q = \bigcup_{Q \in W^{**}} \left( \left( \bigcup_{P \in E(Q)} D(P') \setminus Q \right) \right),$$

and so by (3.18), (3.19), and the disjointness of  $W^{**}$  we have that

$$\begin{aligned} \mu \left( B \cap \left( \bigcup_{Q \in W^*} D(Q') \setminus Q \right) \right) &\leq \sum_{Q \in W^{**}} \mu(Q) \\ &\leq \mu(B(x, CR) \cap X_0) \leq CR^n. \end{aligned}$$

On the other hand,

$$\mu \left( B \cap \left( D(Q'_0) \setminus Q_0 \right) \right) \leq \mu \left( B \cap \left( \bigcup_{Q \in W^*} D(Q') \setminus Q \right) \right).$$

Because clearly  $\mu(B \cap X_0) \leq CR^n$ , (3.17) follows from these last two inequalities.

The proof of the upper bound in (3.13) is now complete. In particular, we have verified assumptions (2.1) - (2.8) in the basic set-up.

Next we verify Assumption A, as described in Section 2. Define  $\mathcal{Q}_0 = \{X_0\}$  and for  $m = 1, 2, \dots$  let  $\mathcal{Q}_m$  be the collection of all flaps  $Q'$  added at the  $m$ th stage to get  $X_m$  from  $X_{m-1}$ . So for  $m = 1, 2, \dots$  we have

$$A_m = X_m \setminus X_{m-1} = \bigcup_{Q' \in \mathcal{Q}_m} (Q' \setminus F_Q).$$

We define  $D_0 = X_0$  and  $D_m = \bigcup_{Q' \in \mathcal{Q}_m} Q'$  for  $m = 1, 2, \dots$ . Note that (2.9) follows from (3.14) and the fact that  $\mu(B^j(y, r)) \geq Cr^n$  for all  $r < R_j(y)$ .

Now assume  $a \in A_\ell$ ,  $y \in A_m$ ,  $m > \ell$ , and that  $a$  and  $y$  lie in the same component of  $X_m \setminus X_{\ell-1}$ . Then there is a sequence of flaps  $Q'_\ell, Q'_{\ell+1}, \dots, Q'_m$  such that  $a \in Q'_\ell$ ,  $y \in Q'_m \setminus F_{Q_m}$ , and that each  $Q'_j$  is a flap at the  $j$ th stage of construction, so that  $Q'_j \subset D_j$  and  $\bigcup_{j=\ell}^m Q'_j$  is connected. (If  $\ell = 0$ , we allow  $X_0$  as a flap.) Let  $\gamma$  be a geodesic joining  $a$  to  $y$ . Choose  $y_\ell = a$ ,  $y_m = y$  and  $y_j \in \gamma \cap Q'_j$  for  $j = \ell + 1, \dots, m - 1$ . For  $j = \ell, \ell + 1$  choose

$$(3.20) \quad r_j = \min\{\ell(Q'_j), d(a, y)\}$$

and for  $j = \ell + 2, \dots, m - 1$  choose  $r_j = \ell(Q'_j)$ . Note that (2.10) follows easily from  $R_j \geq \ell(Q'_j) \geq \ell(Q'_{j+1})$ . Moreover, by (3.20), (2.11) follows if we can show

$$(3.21) \quad \sum_{j=\ell+2}^{m-1} r_j \leq Cd(a, y);$$

in particular, if  $\ell + 2 \geq m$ , there is nothing to prove. Otherwise, each path connecting  $a$  and  $y$  must intersect both  $F_{Q_{\ell+2}}$  and  $\partial Q_{\ell+2}$ ; hence, by (3.12), we have  $r_{\ell+2} \leq Cd(a, y)$ . Estimate (3.21) now follows from the exponential decrease of the  $r'_j$ s for  $j > \ell + 1$ . Finally, (2.12) follows because, for  $j > \ell + 1$ , we have

$$d(y_j, y) \leq \sum_{i=j}^m \text{diam } Q'_i \leq C \text{diam } Q'_j \leq Cr_j.$$

Assumption A is thus verified.

Assumption B is readily verified as follows: Given  $x \in A_k$  and  $y \in A_m$ , take a geodesic  $\gamma$  joining them, and define  $\ell$  to be the smallest integer such that  $\gamma$  meets  $A_\ell$ . Pick  $a$  to be any point in  $A_\ell \cap \gamma$ . Then (2.13) and (2.14) hold; in fact  $d(x, a) + d(a, y) = d(x, y)$ .

After this preparation, we are ready to show that  $X$  admits a Poincaré inequality as in (1.2). Proposition 2.15 is used here.

We shall verify condition (3.2) in  $X$ ; then Theorem 3.1 implies that  $X$  admits a weak Poincaré inequality. Because  $X$  is doubling and geodesic, the main result in [HaK 1] then implies that  $X$  admits a Poincaré inequality as in (1.2). Now (3.2) follows from (2.16) by way of Proposition 2.15. But by Theorem 3.3 each component of  $D_j \cup D_{j+1}$  satisfies a weak Poincaré inequality (with constants independent of  $j$ ), so we infer from the necessity part of Theorem 3.1 that (2.16) indeed holds.

We have thus demonstrated that  $X$  satisfies properties (i) - (v) except compactness. To achieve this last property, we simply consider the completion  $\bar{X}$  of  $X$ . It is easy to see from the construction that  $\bar{X}$  is totally bounded, hence compact, and geodesic. Finally, we extend the measure  $\mu$  to  $\bar{X}$  by setting  $\mu(\bar{X} \setminus X) = 0$ . Upon changing the notation, therefore, we have verified all the properties (i) - (v) in the Introduction.

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