PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 11, Pages 3251–3258
S 0002-9939(00)05510-6

INTEGRAL REPRESENTATION OF LINEAR FUNCTIONALS ON SPACES OF UNBOUNDED FUNCTIONS

PATRIZIA BERTI AND PIETRO RIGO

(Communicated by David R. Larson)

Abstract. Let L be a vector lattice of real functions on a set Ω with $\mathbf{1} \in L$, and let P be a linear positive functional on L. Conditions are given which imply the representation $P(f) = \int f d\pi$, $f \in L$, for some bounded charge π. As an application, for any bounded charge π on a field $\mathcal{F}$, the dual of $L^1(\pi)$ is shown to be isometrically isomorphic to a suitable space of bounded charges on $\mathcal{F}$. In addition, it is proved that, under one more assumption on L, P is the integral with respect to a σ-additive bounded charge.

1. Introduction

Throughout, L is a class of real functions on a set Ω and B is the class of all real bounded functions on Ω. It is assumed that L is a vector lattice including the constants; that is, $\mathbf{1} \in L$ and $af + bg$, $f \vee g$, $f \wedge g$ are in L whenever $f, g \in L$ and $a, b \in \mathbb{R}$. Moreover, $P\colon L \to \mathbb{R}$ is a linear positive functional.

To get integral representations for P is a classical task. Indeed, in addition to the non-recent celebrated results (of Riesz, Daniell, Stone, Radon and many others), there is also recent work in this area; for instance, to prove versions of the Riesz theorem for an arbitrary Hausdorff space, or else to characterize some dual spaces. See [4], [8], [10], [11] and [12]. However, most results concern situations where $L \subset B$ and provide σ-additive integral representations, usually under strong conditions on P. To our knowledge, those cases where $L \not\subset B$, or where the main goal is a finitely additive representation, are almost neglected (an exception is [10]).

In this paper, instead, L is not necessarily a subset of B. In such a framework, conditions are given under which

$$P(f) = \int f d\pi \quad \text{for all } f \in L \tag{1}$$

for some positive bounded charge π. By a *bounded charge*, it is meant a real, bounded, finitely additive measure defined on some field $\mathcal{F}$ of subsets of Ω. A *positive bounded charge* (p.b.c.) is a bounded charge π such that $\pi(A) \geq 0$ for all $A \in \mathcal{F}$. Further, all integrals in this paper are intended in the sense of Dunford and Schwartz; cf. [3] and [6].

Received by the editors August 1, 1997 and, in revised form, December 17, 1998.

2000 *Mathematics Subject Classification.* Primary 28C05; Secondary 60A05.

Key words and phrases. Bounded charge, expectation, integral representation, linear positive functional.

This research was partially supported by M.U.R.S.T. 40% “Processi Stocastici”.

By the Daniell-Stone theorem, if

$$(f_n) \subset L \text{ and } f_n(\omega) \downarrow 0 \text{ for all } \omega \in \Omega \Rightarrow P(f_n) \downarrow 0, \tag{2}$$

then (1) holds for a unique σ-additive p.b.c. π defined on the σ-field generated by the elements of L. Plainly, condition (2) is a form of σ-additivity for the functional P. Thus, one can conjecture that, when (2) is dropped, (1) still holds for some not necessarily σ-additive π. This is not the case, however, as the following simple example shows.

Example 1. Let $\Omega = [0, \infty)$, $f = \sum_{n=1}^{\infty} n I_{[n-1,n)}$ and L be the linear span of f and all the simple functions. After some algebra, it can be checked that $g^+ \in L$ whenever $g \in L$, and thus L is a vector lattice including the constants. Let ν be any p.b.c. on the power set of Ω such that f is ν-integrable, and let $a \in \mathbb{R}$ be such that $a > \int f d\nu$. If $g \in L$, then $g = h + bf$ for some unique simple function h and $b \in \mathbb{R}$. Accordingly, for each $g = h + bf \in L$, define $P(g) = \int h d\nu + ab$. Then, P is linear, positive and (1) fails for any p.b.c. π. To check positivity, fix $g = h + bf \in L$ with $g \geq 0$. Since $g \geq 0$, one has $b \geq 0$, and hence

$$P(g) \geq \int h d\nu + b \int f d\nu = \int g d\nu \geq 0. \quad \square$$

In general, the integral representation of P can fail only on $L - B$. In fact, by the Hahn-Banach theorem, there is a linear positive functional T on B agreeing with P on $L \cap B$. Let $\pi(A) = T(I_A)$ for $A \subset \Omega$. Then, π is a p.b.c. on the power set of Ω and $T(f) = \int f d\pi$ whenever f is a simple function. Since simple functions are dense in B in the sup-norm, $T(f) = \int f d\pi$ for all $f \in B$. Hence, if $\mathcal{F}$ is any field which makes every $f \in L \cap B$ $\pi_{\mathcal{F}}$-integrable, where $\pi_{\mathcal{F}}$ is the restriction of π to $\mathcal{F}$, then $P(f) = T(f) = \int f d\pi_{\mathcal{F}}$ for all $f \in L \cap B$.

Thus, to get an integral representation on all of L, the key point is the possibility of approximating P on $L - B$ through its restriction to $L \cap B$.

The main result of this paper is an equivalent condition for such an approximation, and thus for (1). A sufficient condition for (1), suggested by the proof of the main result, is also obtained. As an application, for any bounded charge π on a field $\mathcal{F}$, the dual of $L^1(\pi)$ is shown to be isometrically isomorphic to $\{\nu \colon \nu$ is a bounded charge on $\mathcal{F}$ and $|\nu| \leq c|\pi|$ for some $c > 0\}$. Moreover, a condition on L is given which implies that (1) holds for a σ-additive p.b.c. π. The condition is just a strengthening of the above-mentioned sufficient condition for (1) (which does not grant σ-additivity of π).

A last note is that the content of this paper is connected with a notion of coherence, for expectations of random variables with values in a Banach space, given in [2]. Other related references are [5] and [9]. Indeed, if L is regarded as a class of real random variables, an expectation on L can be defined as any functional on L which meets some suitable coherence condition. This happens in some approaches to the foundations of probability theory, and in particular in de Finetti's approach. Then, one question is whether a given expectation is an integral with respect to some finitely additive probability measure.

2. Basic definitions and notation

We briefly recall some definitions (see [3] for more information). Given a bounded charge π on a field $\mathcal{F}$ of subsets of Ω, let $\pi^+(A) = \sup\{\pi(A \cap F) \colon F \in \mathcal{F}\}$ for $A \in \mathcal{F}$, $\pi^- = (-\pi)^+$ and $|\pi| = \pi^+ + \pi^-$. Then, π^+, π^- and $|\pi|$ are p.b.c.'s and $\pi = \pi^+ - \pi^-$.

For $A \subset \Omega$, let $|\pi|^*(A) = \inf\{|\pi|(F) \colon A \subset F \in \mathcal{F}\}$. Given an $\mathcal{F}$-simple function $f = \sum_{i=1}^n a_i I_{A_i}$, where $a_1, \ldots, a_n \in \mathbb{R}$ and $\{A_1, \ldots, A_n\}$ is a partition of Ω in $\mathcal{F}$, $\int f d\pi$ is defined as $\int f d\pi := \sum_{i=1}^n a_i \pi(A_i)$. An arbitrary function $f \colon \Omega \to \mathbb{R}$ is π-*integrable* if there is a sequence (f_n) of $\mathcal{F}$-simple functions such that

$$(3) \quad \lim_{n,m} \int |f_n - f_m| d|\pi| = 0 \quad \text{and} \quad \lim_n |\pi|^*(|f_n - f| > \varepsilon) = 0 \quad \text{for each } \varepsilon > 0.$$

If f is π-integrable and (f_n) is as above, $\int f d\pi$ is defined as $\int f d\pi := \lim_n \int f_n d\pi$. Moreover, if (3) holds for some sequence (f_n) of π-integrable (but not necessarily $\mathcal{F}$-simple) functions, then f is π-integrable and $\int |f_n - f| d|\pi| \to 0$.

Let $L^1(\pi)$ be the set of π-integrable functions quotiented by the equivalence relation $f \sim g$ if and only if $|\pi|^*(|f - g| > \varepsilon) = 0$ for each $\varepsilon > 0$. Setting $\|f\|_1 = \int |f| d|\pi|$, $L^1(\pi)$ becomes a normed space. As usual, when confusion does not arise, $L^1(\pi)$ is also regarded as a space of functions (and not of equivalence classes). For instance, we will write $f \in L^1(\pi)$ or $L \subset L^1(\pi)$ to mean that the function f or every element of L, respectively, is π-integrable.

Finally, when Ω is a topological space, $C(\Omega)$ denotes the class of all real continuous functions on Ω.

3. Finitely additive integral representation of functionals

By the discussion in Section 1, there exist a field $\mathcal{F}$ of subsets of Ω and a p.b.c. π on $\mathcal{F}$ such that $L \cap B \subset L^1(\pi)$ and $P(f) = \int f d\pi$ for $f \in L \cap B$. The next result is based on this fact.

Theorem 2. *Let L be a vector lattice of real functions on Ω with $\mathbf{1} \in L$, and let P be a linear positive functional on L. Fix a field $\mathcal{F}$ and a* p.b.c. *π on $\mathcal{F}$ such that $L \cap B \subset L^1(\pi)$ and $P(f) = \int f d\pi$ for $f \in L \cap B$. Then, $L \subset L^1(\pi)$. Moreover, $P(f) = \int f d\pi$ for all $f \in L$ if and only if*

$$(4) \qquad \sup\{P(\phi) \colon 0 \le \phi \le g, \phi \in L\} < \infty \quad \textit{for every } g \in \overline{L} \textit{ with } g \ge 0,$$

where $\overline{L}$ is the closure of L in the $L^1(\pi)$-norm.

Proof. Let $f \in L$. Since $f = f^+ - f^-$, for proving $f \in L^1(\pi)$ it can be assumed $f \ge 0$. Then, since $f \wedge n \in L \cap B$, one has

$$\sup_n \int f \wedge n \, d\pi = \sup_n P(f \wedge n) \le P(f),$$

and this implies $\lim_{n,m} \int |f \wedge n - f \wedge m| d\pi = 0$. For each n, there is a p.b.c. π_n on the power set of Ω such that $\pi_n = \pi$ on $\mathcal{F}$ and $\pi_n(f > n) = \pi^*(f > n)$; cf. [3], Theorem 3.3.3, p. 73. Since π_n extends π, $\int (f \wedge n) d\pi_n = \int (f \wedge n) d\pi$. Hence, given $\varepsilon > 0$,

$$\pi^*(|f \wedge n - f| > \varepsilon) \le \pi_n(f > n) \le \frac{1}{n} \int f \wedge n \, d\pi_n \le \frac{1}{n} P(f) \to 0 \quad \text{as } n \to \infty.$$

It follows that $f \in L^1(\pi)$ and $\int f d\pi = \sup_n \int (f \wedge n) d\pi \le P(f)$. Next, if $P(f) = \int f d\pi$ for all $f \in L$, then $P(\phi) \le \int g d\pi < \infty$ whenever $\phi \in L$, $g \in \overline{L}$ and $0 \le \phi \le g$. Conversely, suppose that $P(f) \ne \int f d\pi$ for some $f \in L$. We prove that, under this assumption, (4) fails. Since $P(f^+) - P(f^-) \ne \int f^+ d\pi - \int f^- d\pi$, it can be assumed that $f \geq 0$. Then, since $P(f) \geq \int f d\pi$, it must be $P(f) > \int f d\pi$. Define a function $g \in \overline{L}$ as follows. Since

$$\sum_{j=1}^{\infty} \int [f \wedge j - f \wedge (j-1)] d\pi = \int f d\pi < \infty,$$

there is a sequence (a_j) such that $a_j > 0$ for all j, $a_j \uparrow \infty$, and

$$\sum_{j=1}^{\infty} a_j \int [f \wedge j - f \wedge (j-1)] d\pi < \infty. \tag{5}$$

Let $g_n = \sum_{j=1}^{n} a_j [f \wedge j - f \wedge (j-1)]$ and $g = \lim_n g_n$. By (5), $\lim_{n,m} \int |g_n - g_m| d\pi = 0$. Moreover, $(g_n) \subset L$ and $\pi^*(|g_n - g| > \varepsilon) \leq \pi^*(f > n) \to 0$ for each $\varepsilon > 0$, so that $g \in \overline{L}$. Observe now that, if $\phi_j := a_{j+1}(f - f \wedge j)$, then $\phi_j \in L$, $0 \leq \phi_j \leq g$, and

$$P(\phi_j) = a_{j+1}\left(P(f) - \int f \wedge j d\pi\right) \geq a_{j+1}\left(P(f) - \int f d\pi\right).$$

Hence, since $P(f) > \int f d\pi$ and $a_j \uparrow \infty$, (4) fails. □

Generally, Theorem 2 helps to decide whether P admits an integral representation; see the next section. Moreover, the proof of Theorem 2 suggests the following criterion.

Theorem 3. *Let L, P and π be as in Theorem 2 and let $\overline{L}$ denote the closure of L in the $L^1(\pi)$-norm (recall that, by Theorem 2, $L \subset L^1(\pi)$). In order that $P(f) = \int f d\pi$ for all $f \in L$, it is sufficient that*

$$g \in \overline{L}, g \geq 0 \text{ and } g \wedge n \in L \text{ for each } n \in \mathbb{N} \Rightarrow g \in L. \tag{i}$$

Proof. Suppose that $P(f) > \int f d\pi$ for some $f \in L$ with $f \geq 0$. Define $(a_j), (g_j), (\phi_j)$ and g as in the second part of the proof of Theorem 2. Then, $g \geq 0$ and $g \in \overline{L}$. Given n, take j with $a_j > n$, and note that $g = g_j$ on $\{f \leq j\}$ and $g \geq g_j \geq a_j > n$ on $\{f > j\}$. Hence, $g \wedge n = g_j \wedge n \in L$. However, since $P(\phi_j) \to \infty$ and $g \geq \phi_j$ for all j, one has $g \notin L$. Thus, (i) fails. □

Condition (i) trivially holds when $L = \overline{L}$. Hence, Theorem 3 yields

$$P(f) = \int f d\pi \quad \text{for all } f \in L \text{ whenever } L \text{ is closed in the } L^1(\pi)\text{-norm.}$$

In particular, one consequence of the above fact is the following. Fix a p.b.c. ν on a field $\mathcal{F}$. If $L = L^1(\nu)$ and P satisfies $P(I_A) = \nu(A)$ for $A \in \mathcal{F}$, then P is the integral with respect to ν; see [2], Theorem (2.13). We now give two more applications of Theorem 3.

Example 4. Let Ω be a topological space. Fix a p.b.c. ν on the field generated by the open sets and define $T(f) = \int f d\nu$ for $f \in C(\Omega) \cap B$. Then, the only extension of T to $L^1(\nu) \cap C(\Omega)$, as a linear positive functional, is the integral with respect to ν. In fact, for any $g \colon \Omega \to \mathbb{R}$ one has $g \in C(\Omega)$ whenever $g \wedge n \in C(\Omega)$ for all n, and thus it suffices to apply Theorem 3 with $L = L^1(\nu) \cap C(\Omega)$ and P such that $P = T$ on $C(\Omega) \cap B$. □

Example 5. Let $\Omega = \mathbb{R}$ and let $\mathcal{D}$ be the class of those real functions which are almost everywhere differentiable (with respect to Lebesgue measure). Given $g\colon \mathbb{R} \to \mathbb{R}$, it can be checked that $|g| \in \mathcal{D}$ whenever $g \in \mathcal{D}$ (and thus $\mathcal{D}$ is a vector lattice), and that $g \in \mathcal{D}$ whenever $g \wedge n \in \mathcal{D}$ for all n. Fix a p.b.c. ν on the field $\mathcal{F}$ generated by the intervals and define $T(I_A) = \nu(A)$ for $A \in \mathcal{F}$. Then, the only extension of T to $L^1(\nu) \cap \mathcal{D}$, as a linear positive functional, is the integral with respect to ν. This follows by Theorem 3, by setting $L = L^1(\nu) \cap \mathcal{D}$ and by taking P such that $P(I_A) = \nu(A)$ for $A \in \mathcal{F}$. $\square$

Clearly, Theorems 2 and 3 also apply to those functionals $T\colon L \to \mathbb{R}$ which admit the representation $T = T_1 - T_2$ with T_1 and T_2 linear and positive. For later purposes, we recall standard conditions on T which are equivalent to such a representation.

Lemma 6. *Let $T\colon L \to \mathbb{R}$. Then, T has the representation $T = T_1 - T_2$, where T_1 and T_2 are linear and positive, if and only if T is linear and*

$$\alpha(f) := \sup\{T(\phi)\colon 0 \leq \phi \leq f, \phi \in L\} < \infty \quad \textit{for every } f \in L \textit{ with } f \geq 0.$$

Moreover, T_1 can be taken as $T_1(f) = \alpha(f^+) - \alpha(f^-)$ for every $f \in L$.

4. Two related results

Let π be a bounded charge on a field $\mathcal{F}$ of subsets of Ω, let

$$\mathcal{B}(\pi) = \{\nu\colon \nu \text{ is a bounded charge on } \mathcal{F} \text{ and } |\nu| \leq c|\pi| \text{ for some } c > 0\},$$

and let us define a norm on $\mathcal{B}(\pi)$ as

$$\|\nu\| = \inf\{c > 0\colon |\nu| \leq c|\pi|\} \quad \text{for } \nu \in \mathcal{B}(\pi).$$

Moreover, let $\mathcal{T}(\pi)$ be the dual of $L^1(\pi)$, with the usual norm $\|T\| = \sup\{|T(f)| : f \in L^1(\pi), \|f\|_1 = 1\}$, $T \in \mathcal{T}(\pi)$. As an application of the material in Section 3, we now prove that $\mathcal{B}(\pi)$ is isometrically isomorphic to $\mathcal{T}(\pi)$.

Theorem 7. *Let $I(\nu)(f) = \int f d\nu$, where $\nu \in \mathcal{B}(\pi)$ and $f \in L^1(\pi)$. Then, $I\colon \mathcal{B}(\pi) \to \mathcal{T}(\pi)$ is an isometric isomorphism.*

Proof. Let $\nu \in \mathcal{B}(\pi)$. Since $L^1(\pi) \subset L^1(\nu)$, $I(\nu)(f)$ is well defined and $|I(\nu)(f)| \leq \|\nu\| \|f\|_1$ for each $f \in L^1(\pi)$. Thus, $I(\nu) \in \mathcal{T}(\pi)$ and $\|I(\nu)\| \leq \|\nu\|$. To prove $\|I(\nu)\| \geq \|\nu\|$, it can be assumed $\nu \neq 0$. Then, given $\varepsilon > 0$, there is $A \in \mathcal{F}$ with

$$|\nu|(A) > (\|\nu\| - \varepsilon)|\pi|(A) \quad \text{and} \quad |\pi|(A) > 0.$$

Moreover, by [3], Theorem 2.6.2, p. 56, there is $E \in \mathcal{F}$ with

$$\nu(E \cap F) > -\varepsilon|\pi|(A) \quad \text{and} \quad \nu(E^c \cap F) < \varepsilon|\pi|(A) \quad \text{for each } F \in \mathcal{F}.$$

Let $f = (I_A I_E - I_A I_{E^c})/|\pi|(A)$. Then, $\|f\|_1 = 1$ and

$$\begin{aligned} I(\nu)(f)|\pi|(A) &= \nu(A \cap E) - \nu(A \cap E^c) = |\nu|(A) - 2(\nu^+(A \cap E^c) + \nu^-(A \cap E)) \\ &> (\|\nu\| - \varepsilon)|\pi|(A) - 4\varepsilon|\pi|(A). \end{aligned}$$

It follows that $\|I(\nu)\| > \|\nu\| - 5\varepsilon$, and thus I is an isometry. Since I is clearly linear, it remains only to prove that I is surjective. Fix $T \in \mathcal{T}(\pi)$. By using Lemma 6, it is easily seen that $T = T_1 - T_2$, where T_1 and T_2 are linear, positive and continuous in the $L^1(\pi)$-norm. Hence, it can be assumed that T is positive. Let λ be a p.b.c.

on the power set of Ω such that $T(f) = \int f d\lambda$ for $f \in L^1(\pi) \cap B$, and let ν be the restriction of λ to $\mathcal{F}$. Then,

$$\nu(F) = T(I_F) \le \|T\| \|I_F\|_1 = \|T\| |\pi|(F) \quad \text{for all } F \in \mathcal{F}.$$

Thus, $\nu \in \mathcal{B}(\pi)$, and this implies $L^1(\pi) \subset L^1(\nu)$ and $T(f) = \int f d\nu$ for $f \in L^1(\pi) \cap B$. Let $0 \le \phi \le g$, where $\phi \in L^1(\pi)$ and g is in the closure of $L^1(\pi)$ in the $L^1(\nu)$-norm. Then

$$T(\phi) = \lim_n T(\phi \wedge n) = \lim_n \int \phi \wedge n \, d\nu \le \int g \, d\nu$$

where the first equality depends on continuity of T in the $L^1(\pi)$-norm. By Theorem 2 (applied with $L = L^1(\pi)$ and $P = T$), it follows that $T = I(\nu)$. $\square$

It is worth noting that, given $\nu \in \mathcal{B}(\pi)$, $I(\nu)$ need not be of the form $I(\nu)(f) = \int fg d\pi$ for each $f \in L^1(\pi)$ and some π-essentially bounded $g \in \Omega \to \mathbb{R}$. In fact, ν can fail to have a density with respect to π. In general, to decide whether $I(\nu)$ has the above form, one needs some finitely additive version of the Radon-Nikodym theorem. See, for instance, [1].

Let us turn now to the second result in this section. Next, Theorem 8 states that, for P to admit a σ-additive integral representation, it is enough that L meets a suitable strengthening of condition (i) in Theorem 3.

Theorem 8. *Let L be a vector lattice with $\mathbf{1} \in L$, and let P be a linear positive functional on L. If*

(ii) $$g \ge 0 \text{ and } g \wedge n \in L \text{ for each } n \in \mathbb{N} \Rightarrow g \in L,$$

then there is a unique σ-additive p.b.c. π, defined on the σ-field generated by the elements of L, such that $L \subset L^1(\pi)$ and $P(f) = \int f d\pi$ for all $f \in L$.

Theorem 8 includes some known results as particular cases. For instance, when Ω is a topological space and $L = C(\Omega)$, Theorem 8 reduces to a result of Hewitt; cf. [7], Theorems 14 and 15. Likewise, when L is the class of all real functions on Ω which are measurable with respect to a given σ-field, Theorem 8 has been obtained by Dubins; cf. [5], Lemmas 1–4 and Theorem 1. On the other hand, in addition to unifying the previous results, Theorem 8 has the merit of giving a general criterion for proving a σ-additive integral representation. As an example, when $\Omega = \mathbb{R}$, Theorem 8 works for $L = \mathcal{D}$ or $L = \mathcal{D} \cap C(\mathbb{R})$, where $\mathcal{D}$ is the class of real functions which are almost everywhere differentiable (with respect to Lebesgue measure); cf. Example 5.

It is possible to give a proof of Theorem 8 based on Theorem 3 only. However, since π is to be σ-additive, it is more convenient to use the result of Daniell and Stone mentioned in Section 1. In fact, if we can prove condition (2), then Theorem 8 automatically follows. Let us fix a p.b.c. ν on the power set of Ω such that $P(f) = \int f d\nu$ for $f \in L \cap B$. In order to prove (2) a useful fact is that, under (ii), Theorem 3 implies

$$L \subset L^1(\nu) \quad \text{and} \quad P(f) = \int f \, d\nu \quad \text{for all } f \in L.$$

We also need two lemmas. In both, L_u denotes the class of those $f \in \Omega \to \mathbb{R}$ such that, for some $(f_n) \subset L$, $f_n \to f$ uniformly.

Lemma 9. *Let* $E = \{\phi \le a\}$ *and* $F = \{\phi \le b\}$*, where* $\phi \in L_u$ *and* $a < b$*. Given* $f \in L \cap B$ *with* $f \ge 0$ *and a scalar* $c \ge \sup f$*, there is* $g \in L \cap B$ *such that*

$$g \ge f, \qquad g = f \quad \text{on } E, \qquad g = c \quad \text{on } F^c.$$

Proof. Take $\phi_0 \in L$ with $|\phi(\omega) - \phi_0(\omega)| < (b-a)/3$ for all $\omega \in \Omega$, and define

$$\psi = \left(\frac{3\phi_0 - 2a - b}{b - a}\right)^+ \wedge 1,$$

and $g = f \vee (c\psi)$. □

Lemma 10. *Let* $A_n = \{\phi \le a_n\}$*,* $n \in \mathbb{N}$*, where* $\phi \in L_u$ *and* $a_1 < a_2 < \cdots$*. If* L *meets* (ii) *and* $A_n \uparrow \Omega$*, then* $\nu(A_n^c) = 0$ *for some* n*.*

Proof. It can be assumed that $A_1 \ne \varnothing$ and $A_n \ne A_{n-1}$ for $n > 1$. Towards a contradiction, suppose that $\nu(A_n^c) > 0$ for all n. By Lemma 9, there is $(g_n) \subset L \cap B$ such that $g_1 = \mathbf{1}$ and, for $n > 1$

$$g_n \ge g_{n-1}, \qquad g_n = g_{n-1} \quad \text{on } A_{n-1}, \qquad g_n = c_n \quad \text{on } A_n^c,$$

where $c_n = n\nu(\Omega)(1 + \sup g_{n-1})/\nu(A_n^c)$. Let $F_1 = A_1$, $F_n = A_n - A_{n-1}$ for $n > 1$, and $g = \sum_{n=1}^\infty I_{F_n} g_n$. Then, for each $n > 1$, $g = g_n$ on A_n and $g \ge g_n = c_n > n$ on A_n^c, and thus $g \wedge n = g_n \wedge n \in L$. By (ii), $g \in L$ and thus $g \in L^1(\nu)$. This leads to a contradiction. In fact,

$$c_n \nu(g \ge c_n) \ge c_n \nu(A_n^c) > n\nu(\Omega) \quad \text{for all } n > 1,$$

and thus $g \notin L^1(\nu)$. □

Proof of Theorem 8. It is enough to prove condition (2). Fix $(f_n) \subset L$ such that $f_n \downarrow 0$. Given $\varepsilon > 0$, it suffices to show that $\nu(f_j \ge \varepsilon) = 0$ for some j. In this case, in fact,

$$0 \le P(f_n) \le P(f_j) = \int f_j d\nu \le \varepsilon \nu(\Omega) \quad \text{for all } n \ge j.$$

Define $\phi_n = (f_n/\varepsilon) \wedge 1$, $\phi = \sum_{n=1}^\infty 2^{-n}\phi_n$ and $A_n = \{\phi \le (n-1)/n\}$. Then, $\phi \in L_u$. Since $f_n \downarrow 0$, $\phi(\omega) < 1$ for all $\omega \in \Omega$, and thus $A_n \uparrow \Omega$. By Lemma 10, $\nu(A_n^c) = 0$ for some n. To conclude the proof it suffices to note that, for some j, $\{f_j \ge \varepsilon\} \subset A_n^c$. □

Acknowledgements

We are grateful to Giorgio Letta for his helpful comments.

References

[1] P. Berti, E. Regazzini, and P. Rigo (1992), *Finitely additive Radon-Nikodym theorem and concentration function of a probability with respect to a probability*, Proc. Amer. Math. Soc. **114**, 1069–1078. MR **92g:**60006

[2] P. Berti, E. Regazzini, and P. Rigo (1994), *Coherent prevision of random elements*, Quaderno I.A.M.I. 94.15, Milano.

[3] K. P. S. Bhaskara Rao and M. Bhaskara Rao (1983), *Theory of charges*, Academic Press, London.

[4] S. T. L. Choy and J. C. S. Wong (1994), *A characterization of the second dual of* $C_0(S, A)$, Proc. Amer. Math. Soc. **120**, 203–211. MR **94b:**46057

[5] L. E. Dubins (1977), *On everywhere-defined integrals*, Trans. Amer. Math. Soc. **232**, 187–194. MR **56:**8783

[6] N. Dunford and J.T. Schwartz (1958), *Linear operators, Part* I, *General theory*, Wiley, Interscience, New York. MR **90g:**47001a

[7] E. Hewitt (1950), *Linear functionals on spaces of continuous functions*, Fund. Math. **37**, 161–189. MR **13:**147g

[8] H. J. E. Konig (1995), *The Daniell-Stone-Riesz representation theorem: Operator theory in function spaces and Banach lattices*, Operator Theory: Adv. Appl., Birkhauser **75**, 191–222. MR **96f:**28016

[9] L. H. Loomis (1954), *Linear functionals and content*, Amer. J. Math. **76**, 168–182. MR **15:**631d

[10] M. Nowak (1992), *Singular linear functionals on non-locally convex Orlicz spaces*, Indag. Mathem., N.S. **3** (3), 337–351. MR **94a:**46034

[11] V. K. Zakharov and A. V. Mikhalev (1997), *Radon problem for regular measures on an arbitrary Hausdorff space*, Fundam. Prikl. Mat. **3**, 801–808 (in Russian).

[12] V. K. Zakharov and A. V. Mikhalev (1998), *The problem of integral representation for Radon measures on a Hausdorff space*, Doklady Mathematics **57**, 337–339. (Translated from Doklady Akademii Nauk 1998, **360**, 13–15). CMP 99:09

Dipartimento di Matematica Pura ed Applicata "G. Vitali", Università di Modena, via Campi 213/B, 41100 Modena, Italy
E-mail address: `berti.patrizia@unimo.it`

Dipartimento di Statistica "G. Parenti", Università di Firenze, viale Morgagni 59, 50134 Firenze, Italy
E-mail address: `rigo@ds.unifi.it`