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FINITE REPRESENTABILITY OF OPERATORS IN THE SENSE OF BELLENOT

MANUELA BASALLOTE AND SANTIAGO DÍAZ-MADRIGAL

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Abstract. We present several new results about the notion of finite representability of operators introduced by Bellenot.

1. Introduction

The theory of finite representability between Banach spaces has been considered by some authors in the more general context of operator theory. Mainly, three definitions have been proposed for the concept of finite representability between two operators and they are due to Beauzamy [B], Bellenot [Be], and Heinrich [H2]. In this paper we are concerned with Bellenot's definition. We show that Bellenot's notion implies Beauzamy's notion in many situations but that they are far from being equivalent in spite of the line which divides these concepts is quite fine. As was pointed out in [Be], there is no kind of relation between Bellenot's and Heinrich's concepts.

Bellenot's opinion about his definition was not very enthusiastic ("This is a strange definition, but it is what ..." [Be, p.7]). However, we present several results which show that, in our opinion, this is a useful and natural concept. Namely, we relate it with conmutative diagrams, we use it to characterize some operator ideals and we connect it with classical topics of linear algebra. We finish the paper giving an alternative proof of one of the main results of [Be, p.7]; that is, the biadjoint of every operator T is finitely representable in T. We would like to mention that there seems to be a gap in the proof given there (see the proof of Condition (D) of Theorem 5 in [Be, p.7]) but, anyway, as we have commented above, our approach is completely different.

In what follows, an operator will always mean a linear and continuous map. An operator $T : X \to Y$ between two Banach spaces is said to be an *ε-isometry* if

$$\left|\|T(x)\| - \|x\|\right| \leq \varepsilon \|x\|, \text{ for all } x \in X,$$

and T is said to be a $(1+\varepsilon)$-*isomorphism* if T is an isomorphism onto its image with $\|T\| \left\|T^{-1}\right\| \leq 1+\varepsilon$.

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The rest of our terminology and notations are quite standard and we refer the reader to [H1] for the theory of ultrapowers and to [Be] and [Di] for the theory of Banach spaces.

2. Finite representability of operators

We begin by recalling the original definition of finite representability of operators introduced by Bellenot [Be].

Definition 2.1. An operator $T_0 : X_0 \longrightarrow Y_0$ is said to be finitely representable in an operator $T : X \longrightarrow Y$ if, for every finite-dimensional subspace $M_0 \subset X_0$ and $\varepsilon > 0$, there is a linear map $V : M_0 \to X$ such that

1. V is an ε-isometry.
2. $|\|T_0(x)\| - \|TV(x)\|| \leq \varepsilon \|x\|$, for all $x \in M_0$.

Bellenot gave an additional third condition but he also noticed that it could be dropped. Recently, the authors (jointly with M. D. Contreras) have shown the following characterization of this notion [BCD].

Theorem 2.1. *Let* $T_0 : X_0 \to Y_0$, $T : X \to Y$ *be two operators and consider the following statements:*

(1) T_0 *is finitely representable in* T.

(2) *There exist an ultrafilter* $\mathcal{U}$ *and two isometries* $\mathcal{J}_1 : X_0 \to (X)_{\mathcal{U}}$, $\mathcal{J}_2 : T_0(X_0) \to (Y)_{\mathcal{U}}$ *such that* $\mathcal{J}_2 T_0 = (T)_{\mathcal{U}} \mathcal{J}_1$.

(3) *For every finite-dimensional subspace* $M_0 \subset X_0$ *and* $\varepsilon > 0$, *there exist a finite-dimensional subspace* $M \subset X$ *and two* ε*-isometries* $V : M_0 \to M$ *and* $W : T_0(M_0) \to T(M)$ *such that* $WT_0 = TV$.

Then, (1) $\Leftrightarrow$ (2) *and, if* T_0 *is injective,* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Beauzamy's definition of finite representability of operators appears just replacing the string "ε-isometries" by "$(1+\varepsilon)$-isomorphisms" in condition (3) of the above theorem [B, p. 241]. Hence, Bellenot's concept implies Beauzamy's concept, whenever T_0 is injective. However, we are going to see that the converse is not true. At this point, we would like to stress that the usual computations used to pass from $(1+\varepsilon)$-isomorphisms to ε'-isometries do not work in our case since they break the commutativity of the diagram.

Theorem 2.2. *An operator* $T : X \to Y$ *is compact if and only if every operator finitely representable in* T *is compact.*

Proof. One implication is trivial. To obtain the other one, we are going to show that $(T)_{\mathcal{U}}$ is compact, for every ultrafilter $\mathcal{U}$ over a certain index set I.

Take $y = (T)_{\mathcal{U}}(x_i)_{\mathcal{U}}$, where $(x_i)_{\mathcal{U}} \in (X)_{\mathcal{U}}$ with $\|(x_i)_{\mathcal{U}}\| \leq 1$. Then, we can choose an element of the corresponding class $(x_i)_i \in \ell_\infty(I, X)$ and a set $I_0 \in \mathcal{U}$ such that $\|x_i\| \leq 1$, for all $i \in I_0$. Since T is compact, the following limit exists

$$z := \lim_{\mathcal{U}} T(x_i) \in \overline{T(B_X)},$$

where B_X is the closed unit ball of X. If $\mathcal{S}$ denotes the canonical injection of Y into $(Y)_{\mathcal{U}}$, we have that

$$\|y - \mathcal{S}(z)\| = \|(Tx_i)_{\mathcal{U}} - \mathcal{S}(z)\| = \lim\nolimits_{\mathcal{U}} \|T(x_i) - z\| = 0.$$

Therefore, $\mathcal{S}(z) = y$ and we have shown that $(T)_{\mathcal{U}}$ maps the closed unit ball of $(X)_{\mathcal{U}}$ into the compact set $\mathcal{S}(\overline{T(B_X)})$. That is, $(T)_{\mathcal{U}}$ is compact.

Take an operator $T_0 : X_0 \to Y_0$ finitely representable in the compact operator T. By Theorem 2.1, we can find an ultrafilter $\mathcal{U}$ and two isometries $\mathcal{J}_1 : X_0 \to (X)_{\mathcal{U}}$, $\mathcal{J}_2 : T_0(X_0) \to (Y)_{\mathcal{U}}$ such that $\mathcal{J}_2 T_0 = (T)_{\mathcal{U}} \mathcal{J}_1$. We know that $(T)_{\mathcal{U}}$ is compact. Since $\mathcal{J}_2$ is an isometry, we also have that T_0 is compact. □

Example 2.1. Finite representability of operators in the sense of Beauzamy is a concept strictly wider than finite representability of operators in the sense of Bellenot.

Proof. Consider the Banach space $X := \left(\bigoplus_{n=1}^{\infty} \ell_q^n\right)_1$, that is, the sum of the finite dimensional spaces ℓ_q^n with the ℓ_1-norm ($1 < q < \infty$), and define the map $T : X \to X$ given by

$$(x_n) \in X \longmapsto T(x_n) := \left(\frac{1}{n} x_n\right) \in X.$$

Clearly, T is a well defined operator with $\|T\| \leq 1$. Moreover, T is the limit in the operator norm of the sequence of operators $P_N T$, where P_N is the projection from X onto the first N coordinates. Since $P_N T$ have finite rank, we deduce that T is compact. On the other hand, denote by T_0 the identity operator on the Banach space ℓ_q ($1 < q < \infty$).

We notice that T_0 is not finitely representable in T in the sense of Bellenot. Otherwise, using that T is compact and according to Theorem 2.2, we obtain that T_0 is compact and, therefore, ℓ_q is finite dimensional.

Now, we are going to see that T_0 is finitely representable in T in the sense of Beauzamy. Let $\varepsilon > 0$ and M_0 be a finite-dimensional subspace of ℓ_q. Take $\varepsilon' > 0$ such that $(1 + \varepsilon')(1 - \varepsilon')^{-1} \leq 1 + \varepsilon$ and $0 < \varepsilon' < \min\{1, \varepsilon\}$.

Since ℓ_q is a $\mathcal{L}_q$-space, we can find an ε'-isometry $V_1 : M_0 \to \ell_q^N$, for some $N \in \mathbb{N}$. If J is the canonical injection of ℓ_q^N into $\left(\bigoplus_{n=1}^{\infty} \ell_q^n\right)_1$, we consider $V := JV_1$. Obviously, V is an ε-isometry.

Define $W : T_0(M_0) = M_0 \to X$ by

$$W(m) = (0, 0, ..., 0, \frac{1}{N} V_1(m)\{\text{the } N\text{-th position}\}, 0, ...), \quad m \in M_0.$$

Obviously, W is an isomorphism onto its image and $WT_0 = TV$. Moreover,

$$\|W\| \leq \frac{1}{N} \|V_1\| \leq \frac{1 + \varepsilon'}{N}, \quad \left\|W^{-1}\right\| \leq N \left\|V^{-1}\right\| \leq \frac{N}{1 - \varepsilon'}.$$

Therefore

$$\|W\| \left\|W^{-1}\right\| \leq \frac{1 + \varepsilon'}{1 - \varepsilon'} \leq 1 + \varepsilon$$

and W is a $(1 + \varepsilon)$-isomorphism. □

We have seen that Bellenot's concept can be characterized by commutative diagrams and ultrapowers. The next theorem gives another result of this type. Roughly speaking, the diagram from Theorem 2.1(3) can be considered the "injective diagram" and the next one is the "surjective diagram".

We recall that, if $T : X \to Y$ is a surjective operator, then open(T) denotes the infimum of the constants $M > 0$ such that, for every $y \in Y$, there is $x \in X$ with

$$Tx = y, \text{ and } \|x\| \leq M \|y\|.$$

Theorem 2.3. *Let* $T_0 : X_0 \to Y_0$ *and* $T : X \to Y$ *be operators with* T_0 *surjective. If* T_0^* *is finitely representable in* T^**, then there exist an ultrafilter* $\mathcal{U}$ *and two surjective operators* $Q_1 : (Y)_{\mathcal{U}} \to Y_0^{**}$ *and* $Q_2 : (X)_{\mathcal{U}} \to ((T_0^*(Y_0^*))^*$ *with*

$$\|Q_1\| \leq 1, \ \ \|Q_2\| \leq 1, \ \ open(Q_1) \leq 1, \ \ open(Q_2) \leq 1,$$

and $Q_1(T)_{\mathcal{U}} = T_0^{**}Q_2$.

Proof. Let $\mathcal{U}$ be an ultrafilter dominating the canonical order filter defined on the set $\mathcal{I}$ of all pairs (M_0, ε) with M_0 a finite-dimensional subspace of Y_0^* and $0 < \varepsilon < 1/2$. By Theorem 2.1, for each $i = (M_0^i, \varepsilon_i) \in \mathcal{I}$, there exist a finite-dimensional subspace M^i of Y^* and two ε_i-isometries, $V_i : M_0^i \longrightarrow M^i$ and $W_i : T_0^*(M_0^i) \longrightarrow T^*(M^i)$ such that $W_i T_0^* = T^* V_i$. Now, for each $i \in \mathcal{I}$ and $y_0^* \in Y_0^*$, we define the element of Y^* given by

$$\mathcal{J}_i(y_0^*) = \begin{cases} V_i(y_0^*) & \text{if} \quad y_0^* \in M_0^i \\ 0 & \text{if} \quad y_0^* \notin M_0^i, \end{cases}$$

and consider the following two maps:

$$\mathcal{J}_1 : Y_0^* \to (Y^*)_{\mathcal{U}}, \qquad y_0^* \mapsto \mathcal{J}_1(y_0^*) := (\mathcal{J}_i(y_0^*))_{\mathcal{U}},$$
$$\mathcal{J}_2 : T_0^*(Y_0^*) \to (X^*)_{\mathcal{U}}, \qquad T_0^*(y_0^*) \mapsto \mathcal{J}_2(T_0^*(y_0^*)) := (T^*)_{\mathcal{U}}\mathcal{J}_1(y_0^*) = (T^*\mathcal{J}_i(y_0^*))_{\mathcal{U}}.$$

It is not difficult to see that $\mathcal{J}_1$ and $\mathcal{J}_2$ are (well defined) linear isometries. It is well-known that the Banach spaces $(Y)_{\mathcal{U}}$ and $(X)_{\mathcal{U}}$ can be canonically placed in $((Y^*)_{\mathcal{U}})^*$ and $((X^*)_{\mathcal{U}})^*$, respectively. At this stage, we define Q_1 as the restriction of $\mathcal{J}_1^*$ to $(Y)_{\mathcal{U}}$ and Q_2 as the restriction of $\mathcal{J}_2^*$ to $(X)_{\mathcal{U}}$. Clearly,

$$\|Q_1\| \leq 1, \|Q_2\| \leq 1, \quad \text{and} \quad Q_1(T)_{\mathcal{U}} = T_0^{**}Q_2.$$

Let us show that Q_1 is surjective and open$(Q_1) \leq 1$. Given $y_0^{**} \in Y_0^{**}$ and, for each $i = (M_0^i, \varepsilon_i) \in \mathcal{I}$, we denote the restriction of y_0^{**} to M_0^i by z_i. We have that $z_i \in (M_0^i)^*$ and $(V_i^*)^{-1} z_i \in \left(M^i\right)^*$. Bearing in mind the isometric identification,

$$\left(M^i\right)^* \equiv Y/(M^i)_\perp, \ \text{(where } (M^i)_\perp \text{ is the orthogonal subset of } M \text{ in } Y\text{)},$$

there is $y_i \in Y$ such that

$$y_i + (M^i)_\perp \equiv (V_i^*)^{-1} z_i \quad \text{and} \quad \|y_i\| \leq \left\|(V_i^*)^{-1} z_i\right\| + \varepsilon_i.$$

For each $i \in \mathcal{I}$,

$$\|y_i\| \leq \left\|(V_i^*)^{-1} z_i\right\| + \varepsilon_i \leq \frac{1}{1 - \varepsilon_i} \|y_0^{**}\| + \varepsilon_i \leq 2 \|y_0^{**}\| + 1.$$

Therefore, we can consider the element $(y_i)_{\mathcal{U}} \in (Y)_{\mathcal{U}}$ and we have

$$\|(y_i)_{\mathcal{U}}\| \leq \lim_{\mathcal{U}} \left(\frac{1}{1 - \varepsilon_i} \|y_0^{**}\| + \varepsilon_i \right) \leq \|y_0^{**}\|.$$

Moreover, we are going to see that $Q_1((y_i)_{\mathcal{U}}) = y_0^{**}$. Given $y_0^* \in Y_0^*$ and using the definition of $\mathcal{J}_1$, we have that

$$\langle Q_1((y_i)_{\mathcal{U}}), y_0^* \rangle = \langle (y_i)_{\mathcal{U}}, \mathcal{J}_1(y_0^*) \rangle = \langle (y_i)_{\mathcal{U}}, (\mathcal{J}_i(y_0^*))_{\mathcal{U}} \rangle = \lim_{\mathcal{U}} \langle y_i, \mathcal{J}_i(y_0^*) \rangle.$$

For each $i = (M_0^i, \varepsilon_i) \in \mathcal{I}$, we consider the projection $P_i : Y_0^* \to M_0^i$. It is not difficult to check that

$$\lim_{\mathcal{U}} \langle y_i, \mathcal{J}_i(y_0^*) \rangle = \lim_{\mathcal{U}} \langle y_i, V_i P_i(y_0^*) \rangle.$$

Then, bearing in mind that $V_iP_i(y_0^*) \in M^i$, we obtain that

$$\begin{aligned}\langle Q_1((y_i)_{\mathcal{U}}), y_0^*\rangle &= \lim_{\mathcal{U}}\langle y_i + (M^i)_\perp, V_iP_i(y_0^*)\rangle = \lim_{\mathcal{U}}\langle (V_i^*)^{-1}z_i, V_iP_i(y_0^*)\rangle \\ &= \lim_{\mathcal{U}}\langle z_i, P_i(y_0^*)\rangle = \lim_{\mathcal{U}}\langle y_0^{**}, P_i(y_0^*)\rangle = \langle y_0^{**}, y_0^*\rangle.\end{aligned}$$

It remains to prove that Q_2 is also surjective with $\operatorname{open}(Q_2) \le 1$. Given $x_0^{**} \in (T_0^*(Y_0^*))^*$ and, by Hahn-Banach's theorem, we extend it to the whole space X_0^* with the same norm and we denote this extension by $z_0^{**} \in X_0^{**}$. For each $i = (M_0^i, \varepsilon_i) \in \mathcal{I}$, we also denote the restriction of z_0^{**} to $T_0^*(M_0^i)$ by z_i. We have $(W_i^*)^{-1}z_i \in (T^*(M^i))^*$ and there is $x_i \in X$ such that

$$x_i + (T^*(M^i))_\perp \equiv (W_i^*)^{-1}z_i \quad \text{and} \quad \|x_i\| \le \left\|(W_i^*)^{-1}z_i\right\| + \varepsilon_i$$

($(T^*(M^i))_\perp$ is the orthogonal of $T^*(M^i)$ in X). Then,

$$\|x_i\| \le \left\|(W_i^*)^{-1}z_i\right\| + \varepsilon_i \le \frac{1}{1-\varepsilon_i}\|x_0^{**}\| + \varepsilon_i \le 2\|x_0^{**}\| + 1$$

and we can consider the element $(x_i)_{\mathcal{U}} \in (X)_{\mathcal{U}}$. We notice that

$$\|(x_i)_{\mathcal{U}}\| \le \lim\left(\frac{1}{1-\varepsilon_i}\|x_0^{**}\| + \varepsilon_i\right) = \|x_0^{**}\|.$$

Moreover, we are going to see that $Q_2((x_i)_{\mathcal{U}}) = x_0^{**}$. Given $T_0^*(y_0^*) \in T_0^*(Y_0^*)$, we have that

$$\begin{aligned}\langle Q_2((x_i)_{\mathcal{U}}), T_0^*(y_0^*)\rangle &= \langle (x_i)_{\mathcal{U}}, \mathcal{J}_2T_0^*(y_0^*)\rangle = \langle (x_i)_{\mathcal{U}}, (T^*\mathcal{J}_i(y_0^*))_{\mathcal{U}}\rangle \\ &= \lim_{\mathcal{U}}\langle x_i, T^*\mathcal{J}_i(y_0^*)\rangle = \lim_{\mathcal{U}}\langle x_i, T^*V_iP_i(y_0^*)\rangle.\end{aligned}$$

Bearing in mind that $T^*V_iP_i(y_0^*) \in T^*(M^i)$, we deduce

$$\begin{aligned}\langle Q_2((x_i)_{\mathcal{U}}), T_0^*(y_0^*)\rangle &= \lim_{\mathcal{U}}\langle x_i + (T^*M^i)_\perp, T^*V_iP_i(y_0^*)\rangle \\ &= \lim_{\mathcal{U}}\langle (W_i^*)^{-1}z_i, W_iT_0^*P_i(y_0^*)\rangle = \lim_{\mathcal{U}}\langle z_i, T_0^*P_i(y_0^*)\rangle \\ &= \lim_{\mathcal{U}}\langle z_0^{**}, T_0^*P_i(y_0^*)\rangle = \langle x_0^{**}, T_0^*(y_0^*)\rangle.\end{aligned}$$

□

Beauzamy characterized the class of uniformly convexifying operators through his concept of finite representability. Heinrich's monograph [H2] also pays special attention to this kind of topic. In the same way, we are going to see that several classes of operators can be characterized with Bellenot's concept (see also Theorem 2.2).

We refer the reader to [B] for the definitions of uniformly convexifying, Rademacher cotype and Rademacher type operators. But, we recall that an operator $T : X \to Y$ is said to be tauberian if the kernel of $T + K$ is reflexive, for all compact operators $K : X \to Y$, and supertauberian if the kernel of $T + K$ is super-reflexive, for all compact operators $K : X \to Y$.

Theorem 2.4. *Let $T : X \to Y$ be an operator.*

(1) *Every operator finitely representable in T is weakly compact if and only if T is uniformly convexifying.*
(2) *Every operator finitely representable in T is weakly conditionally compact if and only if T is of Rademacher type.*

(3) *Every operator finitely representable in* T *is unconditionally convergent if and only if* T *is of Rademacher cotype.*

(4) *Every operator finitely representable in* T *is tauberian if and only if* T *is supertauberian.*

Proof. The first two statements are implicitely given in [BCD], so we omit the proof. The third statement follows easily from Theorem 2.1 and the following two facts: an operator is unconditionally convergent if and only if it does not fix a copy of c_0 [Di]; an operator S is of Rademacher cotype if and only if, for every ultrafilter $\mathcal{U}$, $(S)_{\mathcal{U}}$ does not fix a copy of c_0 [BCD].

(4)($\Rightarrow$) Assume that $T_0 : X_0 \to Y_0$ is finitely representable in T and suppose that T_0 is not tauberian. By [GA], there is $0 < \varepsilon < 1$ such that, for every $\delta > 0$ and every $n \in \mathbb{N}$, we can find $x_0(1), ..., x_0(n)$ in the closed unit ball of X_0 and $x_0^*(1), ..., x_0^*(n)$ in the closed unit ball of X_0^* such that

$$\begin{array}{ll} \langle x_0^*(k), x_0(l)\rangle > \varepsilon & \text{if} \quad 1 \leq k \leq l \leq n, \\ \langle x_0^*(k), x_0(l)\rangle = 0 & \text{if} \quad 1 \leq l < k \leq n, \\ \|T_0(x_0(k))\| < \delta, & \qquad k = 1, ..., n. \end{array}$$

Consider the finite-dimensional subspace of X_0

$$M_0 := \operatorname{span}\{x_0(1), ..., x_0(n)\}.$$

Since T_0 is finitely representable in T, by Theorem 2.1, there exist finite-dimensional subspace M of X and a δ-isometry $V\colon M_0 \to M$ such that $\|TV(x)\| \leq \|T_0(x)\| + \delta\|x\|$. Let us consider $V' := \frac{1}{\|V\|}V$. For each $k = 1, ..., n$, we define the linear functional $y_k^* : M \to \mathbb{K}$ by

$$y_k^*(x) := \left\langle x_0^*(k), (V')^{-1}(x)\right\rangle, \quad x \in M.$$

We have

$$\|y_k^*\| \leq \left\|(V')^{-1}\right\| \|x_0^*(k)\| \leq \frac{1+\delta}{1-\delta} \leq 3.$$

By Hahn-Banach's theorem, we extend y_k^* to X with the same norm and we still denote this extension by y_k^*.

Now, for each $n \in \mathbb{N}$ and $\delta > 0$, we consider the following two finite sets:

$$A(n,\delta) := \{V'(x_0(1)), ..., V'(x_0(n))\}, \quad B(n,\delta) := \left\{\frac{1}{3}y_1^*, ..., \frac{1}{3}y_n^*\right\}.$$

Clearly, $A(n,\delta)$ and $B(n,\delta)$ are contained in the closed unit ball of X and X^*, respectively. Moreover, we have that, if $1 \leq k \leq l \leq n$,

$$\left\langle \frac{1}{3}y_k^*, V'(x_0(l))\right\rangle = \frac{1}{3}\langle x_0^*(k), x_0(l)\rangle > \frac{\varepsilon}{3},$$

if $1 \leq l < k \leq n$,

$$\left\langle \frac{1}{3}y_k^*, V'(x_0(l))\right\rangle = \frac{1}{3}\langle x_0^*(k), x_0(l)\rangle = 0,$$

and, for each $k = 1, ..., n$,

$$\|TV'(x_0(k))\| = \frac{1}{\|V\|}(\|T_0(x_0(k))\| + \delta) \leq \frac{2\delta}{1-\delta}.$$

According to [GA], T is not supertauberian and we get a contradiction.

(4)($\Leftarrow$) Let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$. By Theorem 2.1, we see that $(T)_{\mathcal{U}}$ is finitely representable in T, so $(T)_{\mathcal{U}}$ is a tauberian operator. Then, by [GA], we conclude that T is supertauberian. $\square$

Our next result shows that there are cases where Bellenot's definition can be rewritten in very classical terms. Namely, the corollary to the next proposition is a version of the well-known "singular value decomposition theorem" of linear algebra.

Proposition 2.5. *Let A and B be real matrices, both of dimensions $k \times n$, viewed as operators from ℓ_2^n into ℓ_2^k. Then, A is finitely representable in B if and only if there are two orthogonal matrices P and Q such that $B = PAQ^t$.*

Proof. ($\Leftarrow$) Given a finite-dimensional subspace M_0 of ℓ_2^n and $\varepsilon > 0$, we define V (resp. W) as the restriction to M_0 (resp. to $A(M_0)$) of the isometry induced in ℓ_2^n (resp. in ℓ_2^k) by the matrix P (resp. Q). It is trivial that V and W are $(1+\varepsilon)$-isometries and $VA = BW$. Therefore, by Theorem 2.1, A is finitely representable in B.

($\Rightarrow$) In this implication, we denote by T_A and T_B the operators from ℓ_2^n into ℓ_2^k induced by A and B, respectively. We may assume that A is not the null matrix, so the dimension of $T_A(\ell_2^n)$ is positive. Since T_A is finitely represented in T_B and again, by Theorem 2.1, for each natural number $m > 2$, we can find two $\frac{1}{m}$-isometries

$$V_m : \ell_2^n \to \ell_2^n, \quad W_m : T_A(\ell_2^n) \to T_B(\ell_2^n)$$

such that $W_m T_A = T_B V_m$. We notice that this equality says that W_m is surjective, for all m.

Obviously, (V_m) is a bounded sequence of the Banach space $B(\ell_2^n)$ of all operators from ℓ_2^n into ℓ_2^n. Since $B(\ell_2^n)$ is a finite-dimensional Banach space, we can extract a subsequence (V_{m_l}) of (V_m) such that the sequence (V_{m_l}) converges to some $V \in B(\ell_2^n)$. Clearly, V is an isometry in ℓ_2^n. By a similar argument, we may, and do, assume that the sequence (W_{m_l}) converges in the operator norm to some (surjective) isometry $W : T_A(\ell_2^n) \to T_B(\ell_2^n)$.

Take an orthonormal basis $\{f_1, ..., f_r\}$ $(1 \leq r \leq k)$ of $T_A(\ell_2^n) \subset \ell_2^k$. Since W is a surjective isometry, $\{W(f_1), ..., W(f_r)\}$ is also an orthonormal basis of $T_B(\ell_2^n)$. We extend these bases to the following two orthonormal bases of ℓ_2^k,

$$B_1 := \{f_1, ..., f_r, f_{r+1}, ..., f_k\} \text{ and } B_2 := \{W(f_1), ..., W(f_r), g_{r+1}, ..., g_k\}.$$

Then, we define the linear map $\widetilde{W} : \ell_2^k \to \ell_2^k$ by

$$\sum_{i=1}^{k} \lambda_i f_i \longmapsto \widetilde{W}\left(\sum_{i=1}^{k} \lambda_i f_i\right) := W\left(\sum_{i=1}^{r} \lambda_i f_i\right) + \sum_{i=r+1}^{k} \lambda_i g_i.$$

It is clear that we still have $\widetilde{W} T_A = T_B V$.

At this point, let C_n and C_k be the canonical basis of ℓ_2^n and ℓ_2^k, respectively. It is well known that A and B are the matrices which represent T_A and T_B with respect to C_n-C_k. On the other hand, let Q be the matrix associated to V with respect to C_n-C_n. Since V is an isometry in ℓ_2^n, we have that Q is an orthogonal matrix of order n. By elementary linear algebra, we deduce that $BQ = PA$, where P is the matrix associated to $\widetilde{W}$ with respect to $C_k - C_k$.

Now, let id be the identity in ℓ_2^k, P_1 be the matrix associated to id with respect to $C_k - B_1$ and P_2 be the matrix associated to id with respect to $B_2 - C_k$. Therefore,

$$P = P_2[\text{Identity matrix of order } k]P_1.$$

Moreover, since both of P_1 and P_2 are orthogonal, we deduce that P is orthogonal and this concludes the proof. ☐

Corollary 2.6. *Every real square matrix A is finitely representable in the sense of Bellenot in a diagonal matrix whose entries are the singular values of A.*

Finally, we present an alternative proof of one of the basic results of [Be] which can be seen as a variant in operator theory of the famous “principle of local reflexivity”. As we commented in the introduction, there seems to be a gap in the proof given in [Be, p.7].

Theorem 2.7. *The biadjoint of every operator $T : X \longrightarrow Y$ is finitely representable in T.*

Proof. Let M be a finite-dimensional subspace of X^{**} and $\varepsilon > 0$. Fix $\delta > 0$ and consider a $\delta/2$-net $\{m_1, ..., m_n\}$ in the unit sphere of M. For each $k \in \{1, ..., n\}$, take an element x_k^* of the unit sphere of X^* such that $|\langle m_k, x_k^* \rangle| + \delta/2 > 1$. If we define F_0 :=span$\{x_1^*, ..., x_n^*\}$, we have that

$$(1-\delta)\|m\| \leq \sup\{|\langle m, f_0\rangle| : f_0 \in F_0, \|f_0\| = 1\}, \quad \text{for all } m \in M.$$

Now, we consider a $\delta/2$-net $\{h_1, ..., h_r\}$ in the unit sphere of (the finite-dimensional space) $T^{**}(M)$. For each $k \in \{1, ..., r\}$, take an element y_k^* in the unit sphere of Y^* such that $|\langle h_k, y_k^* \rangle| + \delta/2 > 1$. If we define F_1 :=span$\{y_1^*, ..., y_r^*\}$, we have that

$$(1-\delta)\|T^{**}(m)\| \leq \sup\{|\langle T^{**}(m), f_1\rangle| : f_1 \in F_1, \|f_1\| = 1\}, \quad \text{for all } m \in M.$$

Let $B(M, X)$ be the Banach space of all linear and continuous operators from M to X. Given $m \in M$ and $x^* \in X^*$, we denote by $\phi(m, x^*)$ the element of the dual of $B(M, X)$ defined by

$$\langle \phi(m, x^*), T\rangle = \langle T(m), x^*\rangle, \quad T \in B(M, X).$$

Put

$$F := \text{ span}\{\phi(m, f_0), \phi(m, T^* f_1) : m \in M, f_0 \in F_0, f_1 \in F_1\} \subset B(M, X)^*,$$

and let I be the formal inclusion of M into X^{**} viewed as an element of the bidual of $B(M, X)$ [D]. Then, by Helly's lemma and, for each finite-dimensional subspace $G \subset (B(M, X))^*$ containing F, we can find an operator $S_G : M \to X$ such that $\langle g, S_G\rangle = \langle I, g\rangle$, for every $g \in G$ and $\|S_G\| \leq (1+\delta)\|I\| \leq 1+\delta$.

Therefore, for every $m \in M$, we have

$$\begin{aligned} (1-\delta)\|m\| &\leq \sup\{|\langle m, f_0\rangle| : f_0 \in F_0, \|f_0\| = 1\} \\ &= \sup\{|\langle I, \phi(m, f_0)\rangle| : f_0 \in F_0, \|f_0\| = 1\} \\ &= \sup\{|\langle S_G(m), f_0\rangle| : f_0 \in F_0, \|f_0\| = 1\} \\ &\leq \|S_G(m)\| \leq (1+\delta)\|m\|. \end{aligned}$$

That is, S_G is an δ-isometry, for every G. A similar argument shows that, for every G,

$$(1-\delta)\|T^{**}(m)\| \leq \|TS_G(m)\|, \text{ for all } m \in M.$$

That is, we have shown that, for every $\delta > 0$ and every finite-dimensional subspace $G \subset (B(M,X))^*$ containing F, the subset $C_\delta^G \subset B(M,X)$ is not empty, where C_δ^G is formed by all δ-isometries S from M to X satisfying that

$$\langle g, S\rangle = \langle I, g\rangle, \quad g \in G,$$

$$(1-\delta)\left\|T^{**}(m)\right\| \leq \left\|TS(m)\right\|, \quad m \in M.$$

Define $C_\delta := \bigcup\{C_\delta^G : G \subset (B(M,X))^*, G \supset F \text{ and } \dim G < \infty\}$. Looking at the subspaces F_0 and F_1, it is not difficult to see that C_δ is a convex subset of $B(M,X)$. Moreover, let us consider the following subsets of Y^n:

$$D := \left\{(y_1, ..., y_n) \in Y^n : \|y_k\| < \left\|T^{**}(m_k)\right\| + \delta,\ k = 1, ..., n\right\},$$

$$\widetilde{C_\delta} := \left\{(TS(m_1), ..., TS(m_n)) \in Y^n : S \in C_\delta\right\}.$$

Clearly, D is a non-empty convex open subset of Y^n and $\widetilde{C_\delta}$ is a non-empty convex subset of Y^n. We claim that $D \cap \widetilde{C_\delta} \neq \emptyset$. Therefore, there is an δ-isometry $S : M \to X$ such that

$$(1-\delta)\left\|T^{**}(m)\right\| \leq \left\|TS(m)\right\|, \quad \text{for all } m \in M,$$

$$\left\|TS(m_k)\right\| \leq \left\|T^{**}(m_k)\right\| + \delta, \quad k = 1, ..., n.$$

Using that $\{m_1, ..., m_n\}$ is a $\delta/2$-net in the unit sphere of M, we deduce that

$$\left\|TS(m)\right\| \leq \left\|T^{**}(m)\right\| + \delta + \delta\left\|T\right\| + \frac{\delta}{2}(1+\delta)\left\|T\right\|, \quad \text{for all } m \in M.$$

Hence, according to Definition 2.1, if we choose $\delta > 0$ such that

$$\delta + \delta\left\|T\right\| + \frac{\delta}{2}(1+\delta)\left\|T\right\| < \varepsilon \text{ and } \delta < \varepsilon,$$

we have that T^{**} is finitely representable in T in the sense of Bellenot.

Now, we are going to prove the claim. If $D \cap \widetilde{C_\delta} = \emptyset$, by Hahn-Banach's theorem, we can find $\alpha \in \mathbb{R}$ and $g = (g_1, ..., g_n) \in (Y^*)^n \equiv (Y^n)^*$ such that

$$\operatorname{Re}\langle g, y_1\rangle < \alpha \leq \operatorname{Re}\langle g, y_2\rangle, \quad \text{for all } y_1 \in D,\ y_2 \in \widetilde{C_\delta}.$$

We see that $g \neq 0$. Consider the space G formed by the linear hull of the set

$$\left\{\phi(m, f_0), \phi(m, T^*(f_1)), \phi(m, T^*(g_k)) : k = 1, ..., n,\ m \in M, f_0 \in F_0, f_1 \in F_1\right\}.$$

The corresponding operator $S_G \in C_\delta$ satisfies that

$$\begin{aligned} \alpha &\leq \operatorname{Re}\langle g, (TS_G(m_1), ..., TS_G(m_n))\rangle = \operatorname{Re}\left(\sum_{k=1}^{n}\langle T^*(g_k), S_G(m_k)\rangle\right) \\ &= \operatorname{Re}\left(\sum_{k=1}^{n}\langle g_k, T^{**}(m_k)\rangle\right) = \operatorname{Re}\langle g, (T^{**}(m_1), ..., T^{**}(m_n))\rangle. \end{aligned}$$

For each $k = 1, ..., n$ with $T^{**}(m_k) \neq 0$, we take $\widetilde{d_k} \in Y$ with $\left\|\widetilde{d_k}\right\| \leq \left\|T^{**}(m_k)\right\|$ such that

$$\left|\langle\widetilde{d_k}, g_k\rangle - \langle T^{**}(m_k), g_k\rangle\right| < \frac{\delta}{16n}\left\|g\right\|_1.$$

Define, for each $k=1,...,n$,

$$d_k := \begin{cases} 0 & \text{if } \quad T^{**}(m_k)=0, \\ \widetilde{d_k} & \text{if } \quad T^{**}(m_k)\neq 0. \end{cases}$$

Moreover, for each $k=1,...,n$, we choose y_k in the closed unit ball of Y with

$$\operatorname{Re}\langle g_k, y_k\rangle \geq \frac{1}{2}\left\|\operatorname{Re}(g_k)\right\| \geq \frac{1}{4}\left\|g_k\right\|.$$

Hence

$$\begin{aligned} \alpha \quad &\geq \quad \operatorname{Re}\left(\sum_{k=1}^{n}\langle g_k, d_k+\frac{\delta}{2}y_k\rangle\right) \geq \operatorname{Re}\left(\sum_{k=1}^{n}\langle g_k, d_k\rangle\right)+\frac{\delta}{2}\frac{1}{4}\sum_{k=1}^{n}\left\|g_k\right\| \\ &\geq \quad \operatorname{Re}\left(\sum_{k=1}^{n}\langle g_k, T^{**}(m_k)\rangle\right)-n\frac{\delta}{16n}\left\|g\right\|_1+\frac{\delta}{8}\left\|g\right\|_1 \\ &= \quad \operatorname{Re}\langle g,(T^{**}(m_1),...,T^{**}(m_n))\rangle+\frac{\delta}{16}\left\|g\right\|_1 \geq \alpha+\frac{\delta}{16}\left\|g\right\|_1. \end{aligned}$$

Therefore, we have that $0 \geq \left\|g\right\|_1$, so $g=0$ and we obtain a contradiction. ☐

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Departamento de Matemática Aplicada II, Escuela Superior de Ingenieros, Universidad de Sevilla, Camino de los Descubrimientos, 41092 Sevilla, Spain

E-mail address: `mabas@matinc.us.es`

Departamento de Matemática Aplicada II, Escuela Superior de Ingenieros, Universidad de Sevilla, Camino de los Descubrimientos, 41092 Sevilla, Spain

E-mail address: `madrigal@cica.es`