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FINSLER METRICS AND ACTION POTENTIALS

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Abstract. We study the behavior of Mañé's action potential Φ_k associated to a convex superlinear Lagrangian, for k bigger than the critical value $c(L)$. We obtain growth estimates for the action potential as a function of k. We also prove that the action potential can be written as $\Phi_k(x, y) = D_F(x, y) + f(y) - f(x)$ where f is a smooth function and D_F is the distance function associated to a Finsler metric.

1. Introduction

Let M be a closed riemannian manifold with riemannian metric $\langle v, v\rangle$. Consider the mechanical Lagrangian

$$L : TM \to \mathbb{R},$$

$$(x, v) \mapsto \frac{1}{2}\langle v, v\rangle_x - U(x)$$

where $U(x)$ is a differentiable function on M called the potential.

It is well known that, on a fixed level of energy e, bigger than the maximum of U the lagrangian flow is conjugate to the geodesic flow with metric $2(e - U(x))\langle v, v\rangle$. Moreover the reduced action of the Lagrangian is the distance for this metric. Either or both of these statements are known as the Maupertuis principle. See the books [1], [2] or [5].

Consider now a general convex superlinear Lagrangian $L : TM \to \mathbb{R}$. This means that L restricted to each T_xM has positive definite Hessian and

$$\lim_{|v|\to\infty} \frac{L(x, v)}{|v|} = \infty,$$

uniformly on $x \in M$.

It was proven in [4] that for large energy values the lagrangian flow is conjugate to the flow of a Finsler metric. See below for the precise statement. In Theorem 1 we prove a generalization of the other statement of the Maupertuis principle. This was motivated by discussions with R. Montgomery about the presentation in [5], which also motivated Theorem 2.

Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian associated to L and let $\mathcal{L} : TM \to T^*M$ be the Legendre transform $(x, v) \mapsto \partial L/\partial v(x, v)$. Since M is compact, the extremals of L give rise to a complete flow $\varphi_t : TM \to TM$ called the Euler-Lagrange flow of

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the Lagrangian. Using the Legendre transform we can push forward φ_t to obtain another flow φ_t^* which is the Hamiltonian flow of H with respect to the canonical symplectic structure of T^*M. Recall that the *energy* $E : TM \to \mathbb{R}$ is defined by

$$E(x,v) = \frac{\partial L}{\partial v}(x,v).v - L(x,v).$$

Since L is autonomous, E is a first integral of the flow φ_t.

Recall that the action of the Lagrangian L on an absolutely continuous curve $\gamma : [a,b] \to M$ is defined by

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t))\, dt.$$

Given two points x and y in M and $T > 0$ denote by $\mathcal{C}_T(x,y)$ the set of absolutely continuous curves $\gamma : [0,T] \to M$, with $\gamma(0) = x$ and $\gamma(T) = y$. For each $k \in \mathbb{R}$ we define the *action potential* $\Phi_k : M \times M \to \mathbb{R}$ by

$$\Phi_k(x,y) = \inf\{A_{L+k}(\gamma) : \ \gamma \in \bigcup_{T>0} \mathcal{C}_T(x,y)\}.$$

The *critical value of* L, which was introduced by Mañé in [6], is the real number $c(L)$ defined as the infimum of $k \in \mathbb{R}$ such that for some $x \in M$, $\Phi_k(x,x) > -\infty$. For $k \geq c(L)$, we have that $\Phi_k(x,y) > -\infty$ for every x, y and it is a Lipschitz function that satisfies the triangle inequality.

For any $k > c(L)$ the flow on the energy level k is conjugate to the geodesic flow of an appropriately chosen Finsler metric on M (see [4]).

Given a Finsler metric $\sqrt{F}$ and an absolutely continuous curve γ we can define its Finsler length as

$$l_F(\gamma) = \int \sqrt{F(\dot\gamma)}.$$

Observe that since the Finsler metric is homogeneous of degree one, the definition does not depend on the parameterization of the curve. Finally we define the Finsler distance as

$$D_F(x,y) = \inf\{l_F(\gamma)\}$$

where the infimum is taken over all absolutely continuous curves joining x and y.

Theorem 1. *If k is bigger than the critical value, then there exist a Finsler metric $\sqrt{F}$ and a C^∞ real valued function f on M such that $\Phi_k(x,y) = D_F(x,y) + f(y) - f(x)$. Moreover if k is bigger than $-\inf L$, then we can choose $f = 0$.*

As a consequence of Theorem 1 we have that there is a neighborhood V of the diagonal Δ in $M \times M$, such that Φ_k is differentiable in $V \setminus \Delta$.

For x, y fixed and $T > 0$ define

$$S(T) = \inf\{A_L(\gamma) : \ \gamma \in \mathcal{C}_T(x,y)\}.$$

It is easily shown that $S(T)$ is continuous. Although $S(T)$ is not necessarily convex, its Legendre transform:

$$S^*(e) = \max_{T>0}(eT - S(T))$$

is a well defined convex function and coincides with the Legendre transform of the convex hull $\overline{S}$ of S. Notice that

$$\Phi_k(x,y) = -S^*(-k) \tag{1}$$

and so the domain of S^* is $\operatorname{dom} S^* = (-\infty, -c(L)]$. It follows from the definition of the action potential that $g(k) = \Phi_k(x,y)$ is nondecreasing and so is S^*.

Theorem 2. *For all x, y in M we have that:*

(a) *g grows slower than any linear function; that is,*

$$\lim_{k\to\infty} \frac{g(k)}{k} = 0.$$

(b) *The right derivative of g at $c(L)$ is infinite.*

(c) $\lim_{T\to\infty} S(T)/T = -c(L)$.

2. Proofs

Proof of Theorem 2. It is well known that if f is a convex function of a real variable, then

(1) If $x \in \operatorname{int}(\operatorname{dom} f)$, then both one side derivatives exist and $f'_-(x) \le f'_+(x)$.

(2) If $x \in \operatorname{dom} f$ is a boundary point, then the corresponding one side derivative exists.

(3) If $x < y$, $f'_+(x) \le f'_-(y)$.

Define

$$\operatorname{rang} \partial f = \bigcup_{x\in \operatorname{dom} f} [f'_-(x), f'_+(x)].$$

It is proved in [8], Section 24, that

$$\operatorname{int}(\operatorname{dom} f^*) \subset \operatorname{rang} \partial f \subset \operatorname{dom} f^*.$$

Therefore

$$\operatorname{rang} \partial S^* = \operatorname{dom} S^{**} = \operatorname{dom} \overline{S} = (0, \infty).$$

Thus

$$\lim_{e\to -\infty} \frac{S^*(e)}{e} = 0$$

and

$$S^{*\prime}_-(-c(L)) = \lim_{e\to -c(L)} S^{*\prime}_-(e) = \infty.$$

From equation (1), items (a) and (b) follow.

By the same kind of arguments $\lim_{T\to\infty} \overline{S}(T)/T = -c(L)$, and then

$$-c(L) \le \liminf_{T\to\infty} \frac{S(T)}{T}.$$

To prove the other inequality, let μ be an ergodic minimizing probability, that is, an invariant ergodic probability for the lagrangian flow φ_t such that

$$m := \int L\, d\mu \le \int L\, d\nu$$

for any invariant probability ν. Mather proved that such measures exist (see [7]). Let $\theta \in TM$ be a regular point for μ, such that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T L(\varphi_t(\theta))dt = m.$$

Let $\pi : TM \to M$ be the natural projection. Comparing with the curve γ obtained by joining x and $\pi(\theta)$ with a short curve, then following the curve $\pi\varphi_t(\theta)$ and then joining $\pi\varphi_T(\theta)$ and y with a short curve, we have that for any given $\varepsilon > 0$ and T large enough

$$S(T) \le (m+\varepsilon)T + O(1).$$

So

$$\limsup_{T\to\infty} \frac{S(T)}{T} \le (m+\varepsilon).$$

Item (c) now follows from the fact due to Mañé [6, 3] that $m = -c(L)$. □

Proof of Theorem 1. We begin with the last statement. Note that $L + k$ is bigger than zero if and only if $H(x, 0) < k$. Indeed

$$H(x,p) = \max_{v\in T_xM}(pv - L(x,v));$$

then

$$H(x,0) = \max_{v\in T_xM}(-L(x,v)) = -\min_{v\in T_xM}(L(x,v)).$$

So if k is bigger than $-\inf L$, then $H^{-1}(-\infty, k)$ contains the zero section of T^*M.

Now define a new Hamiltonian G on T^*M minus the zero section such that G takes the value μ on $H^{-1}(k)$ and such that $G(x,\lambda p) = \lambda^2 G(x,p)$ for all positive λ. Since G is positively homogeneous of degree two and convex in p, it follows that the Legendre transform F associated to G is the square of a Finsler metric.

Since by definition $G^{-1}(\mu) = H^{-1}(k)$, it follows that the Hamiltonian flows of G and H coincide up to reparameterization on the energy level $G^{-1}(\mu) = H^{-1}(k)$ and therefore the Euler-Lagrange solutions of L with energy k are reparameterizations of geodesics of $\sqrt{F}$.

We claim that for an appropriate choice of μ and if $E(x,v) = k$, then

$$\sqrt{F(x,v)} = L + k.$$

It is proved in [6, 3] that for k greater than the critical value and for any x, y in M there exists γ such that $A_{L+k}(\gamma) = \Phi_k(x,y)$. Moreover γ is a solution of the Euler-Lagrange equation and has energy k. Also, if $k > c(L)$, every curve can be reparameterized to have energy k and the Finsler length does not depend on the reparameterization. By the definitions of both D_F and Φ_k, we may restrict ourselves to curves with energy k and Theorem 1 follows in this case.

Proof of the claim. Since G is homogeneous of degree 2 it follows from Euler's formula that F and G take the same value at Legendre related points.

Define $\lambda(x,p)$ such that $H(x, \frac{p}{\lambda}) = k$; then $G(x,p) = \mu\lambda^2(x,p)$. We have

$$\frac{\partial H}{\partial p}(x, \frac{p}{\lambda})\lambda^{-1} - \frac{\partial H}{\partial p}(x, \frac{p}{\lambda}) \cdot p\, \lambda^{-2}\frac{\partial \lambda}{\partial p} = 0 \tag{2}$$

and

$$\frac{\partial G}{\partial p} = 2\mu\lambda\frac{\partial \lambda}{\partial p}.$$

Multiplying (2) by $2\mu\lambda^3$ we then get

$$\frac{\partial H}{\partial p}(x, \frac{p}{\lambda}) \cdot p \, \frac{\partial G}{\partial p} = 2G(x,p)\frac{\partial H}{\partial p}(x, \frac{p}{\lambda}). \tag{3}$$

Suppose now that $E(x,v) = k$ and let $P(x,v) = \partial L/\partial v$; then by definition we have

$$\begin{aligned} \lambda(x, P(x,v)) &= 1, \\ G(x, P(x,v)) &= \mu, \\ \frac{\partial H}{\partial p}(x, P(x,v)) &= v, \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial H}{\partial p}(x, P(x,v)) \cdot P(x,v) &= v\frac{\partial L}{\partial v} \\ &= L + k \\ &> 0. \end{aligned} \tag{4}$$

Hence from (3) we have

$$\frac{\partial G}{\partial p}(x, P(x,v)) = \frac{2v}{v \cdot P(x,v)}.$$

Since $\partial G/\partial p$ is homogeneous of degree one and from (4) $v \cdot P(x,v)$ is positive, we obtain

$$\frac{\partial G}{\partial p}(x, \frac{1}{2}v \cdot P(x,v)P(x,v)) = v.$$

So v is related to $\frac{1}{2}v \cdot P(x,v)P(x,v)$ with respect to the Legendre transform of F. Hence

$$\begin{aligned} F(x,v) &= G(x, \frac{1}{2}v \cdot P(x,v)P(x,v)) \\ &= \frac{(v \cdot P(x,v))^2}{4}G(x, P(x,v)) \\ &= \frac{(v \cdot P(x,v))^2}{4}\mu. \end{aligned}$$

So if $\mu = 4$,

$$\sqrt{F(x,v)} = v \cdot \frac{\partial L}{\partial v}.$$

Now let k be bigger than $c(L)$. Then by a corollary in [4] there exists a C^∞ real valued function f on M, such that $H(x, df_x) < k$. Define as in [4] $H_{df}(x,p) = H(x, p + df_x)$. The Hamiltonian flows are conjugate by $\psi(x,p) = (x, p - df_x)$. The

Legendre transformation L_{df} of H_{df} is

$$\begin{aligned} L_{df}(x,v) &= \max_{p\in T_x^*M}(pv - H_{df}(x,p)) \\ &= \max_{p\in T_x^*M}(pv - H(x,p+df_x)) \\ &= \max_{p\in T_x^*M}((p-df_x)v - H(x,p)) \\ &= L(x,v) - df_x v. \end{aligned}$$

It turns out that

$$\begin{aligned} E(L_{df}) &= E(L), \\ c(L_{df}) &= c(L), \\ \Phi_k(L_{df})(x,y) &= \Phi_k(L)(x,y) - f(y) + f(x). \end{aligned}$$

So as the zero section is contained in $H_{df}^{-1}(-\infty,k)$, $L_{df}+k$ is positive and there is a Finsler metric such that

$$\Phi_k(L_{df})(x,y) = D_F(x,y).$$

So

$$\Phi_k(L)(x,y) = D_F(x,y) + f(y) - f(x).$$

□

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