

## FINITE DIMENSIONAL INJECTIVE OPERATOR SPACES

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ABSTRACT. We show that finite dimensional injective operator spaces are corners  $p\mathcal{A}p^\perp$  of finite dimensional  $C^*$ -algebras  $\mathcal{A}$ .

An operator space  $\mathcal{E} \subseteq B(H)$  is said to be *injective* if there is a completely contractive projection of  $B(H)$  onto  $\mathcal{E}$ . In [R], Ruan characterized the injective operator spaces  $\mathcal{E}$  as having the form  $p\mathcal{A}q$  for a pair of projections  $p$  and  $q$  in an injective  $C^*$ -algebra  $\mathcal{A}$ . This left open the problem of deciding whether finite dimensional injective operator spaces had such representations for *finite dimensional*  $C^*$ -algebras  $\mathcal{A}$ . This question was posed to the author by Professor David Blecher, and answered, several years ago. Although it is mentioned in [B], the proof was never published. Since there has been some recent interest in this result, [E], [EN], [ER1], [ER2], it now seems appropriate to fill this small gap in the operator space literature.

**Theorem.** *A finite dimensional operator space  $\mathcal{E}$  is injective if and only if it is completely isometrically isomorphic to  $p\mathcal{A}p^\perp$ , where  $\mathcal{A}$  is a finite dimensional  $C^*$ -algebra and  $p \in \mathcal{A}$  is a projection.*

*Proof.* All finite dimensional  $C^*$ -algebras are injective, so one direction is trivial. Suppose that  $\mathcal{E}$  is injective. By [R], there is an injective  $C^*$ -algebra  $\mathcal{A}$  and projections  $p$  and  $q$  so that  $\mathcal{E}$  is completely isometrically isomorphic to  $p\mathcal{A}q$ . A slight modification of Ruan's result allows us to assume that  $q = p^\perp$ . First replace  $\mathcal{A}$  by  $\mathbb{M}_2 \otimes \mathcal{A}$ ,  $p$  by  $e_{11} \otimes p$ , and  $q$  by  $e_{22} \otimes q$ . This identifies  $\mathcal{E}$  with  $e_{12} \otimes \mathcal{E}$ , and allows us to assume that  $p \perp q$ . Then replace  $\mathcal{A}$  by  $(p+q)\mathcal{A}(p+q)$  to obtain  $q = p^\perp$ . These steps preserve injectivity.

Define two  $C^*$ -algebras by  $\mathcal{B} = p\mathcal{A}p$  and  $\mathcal{C} = p^\perp\mathcal{A}p^\perp$ . Then, relative to  $p$  and  $p^\perp$ ,  $\mathcal{A}$  can be represented by matrices

$$\left\{ \begin{pmatrix} b & e \\ f^* & c \end{pmatrix} : b \in \mathcal{B}, c \in \mathcal{C}, e, f \in \mathcal{E} \right\}.$$

Now  $\mathcal{E}$  is a  $(\mathcal{B}, \mathcal{C})$ -bimodule, so we may define bounded maps

$$\lambda : \mathcal{B} \rightarrow B(\mathcal{E}), \quad \rho : \mathcal{C} \rightarrow B(\mathcal{E})$$

by

$$\lambda(b)(e) = be, \quad \rho(c)(e) = ec, \quad b \in \mathcal{B}, c \in \mathcal{C}, e \in \mathcal{E}.$$

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The first is a homomorphism and the second an anti-homomorphism. Thus their respective kernels  $\mathcal{J} \subseteq \mathcal{B}$  and  $\mathcal{K} \subseteq \mathcal{C}$  are closed two-sided ideals, which are necessarily self-adjoint. Since  $\mathcal{E}$  is finite dimensional, it follows that the quotient  $C^*$ -algebras  $\mathcal{B}/\mathcal{J}$  and  $\mathcal{C}/\mathcal{K}$  also have this property. If  $j \in \mathcal{J}$ ,  $k \in \mathcal{K}$  and  $f \in \mathcal{E}$ , then

$$f^*j = (j^*f)^* = 0, \quad kf^* = (fk^*)^* = 0.$$

It follows that

$$\begin{pmatrix} j & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} b & e \\ f^* & c \end{pmatrix} = \begin{pmatrix} jb & 0 \\ 0 & kc \end{pmatrix}$$

and

$$\begin{pmatrix} b & e \\ f^* & c \end{pmatrix} \begin{pmatrix} j & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} bj & 0 \\ 0 & ck \end{pmatrix},$$

for  $b \in \mathcal{B}$ ,  $c \in \mathcal{C}$ ,  $j \in \mathcal{J}$ ,  $k \in \mathcal{K}$ , and  $e, f \in \mathcal{E}$ . These equations show that  $\mathcal{I} \equiv \mathcal{J} \oplus \mathcal{K}$  is a two-sided norm closed ideal in  $\mathcal{A}$ . Let  $\pi : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$  be the quotient map, which is a complete contraction. The inequality

$$\|E\| = \left\| \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} J & E \\ 0 & K \end{pmatrix} \right\|$$

for all  $E \in \mathbb{M}_n(\mathcal{E})$ ,  $J \in \mathbb{M}_n(\mathcal{J})$ , and  $K \in \mathbb{M}_n(\mathcal{K})$  implies that  $\pi$  implements a completely isometric isomorphism between  $\mathcal{E}$  and  $\pi(\mathcal{E}) \subseteq \mathcal{A}/\mathcal{I}$ , and this last  $C^*$ -algebra is finite dimensional. The proof is completed by observing that  $\pi(\mathcal{E}) = \pi(p)(\mathcal{A}/\mathcal{I})\pi(p)^\perp$ .  $\square$

It is well known that a general finite dimensional operator space cannot be embedded in a finite dimensional  $C^*$ -algebra (since otherwise an easy argument would show erroneously that  $B(H)$  is nuclear) and so the hypothesis of *injectivity* in the theorem is not superfluous. It should also be noted that, as pointed out by Professor Marius Junge, the finite dimensional injective operator spaces form the correct class for defining the  $\mathcal{OL}_p$  spaces of [ER1].

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