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PROJECTIVE BOUNDEDNESS AND CONVOLUTION OF FRÉCHET MEASURES

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ABSTRACT. Fréchet measures of order n ($\mathcal{F}_n$-measures) are the measure-theoretic analogues of bounded n-linear forms on products of $C_0(K)$ spaces. In an LCA setting, convolution of $\mathcal{F}_2$-measures is always defined, while there exist $\mathcal{F}_3$-measures whose convolution cannot be defined. In a three-dimensional setting, we demonstrate the existence of an $\mathcal{F}_2$-measure which cannot be convolved with arbitrary $\mathcal{F}_3$-measures.

Let $(\mathcal{X}_1, \mathcal{A}_1), \ldots, (\mathcal{X}_n, \mathcal{A}_n)$ be measurable spaces. A scalar-valued function μ on $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ is a Fréchet measure (an $\mathcal{F}_n$-measure) if, when any $n-1$ coordinates are fixed, μ is a measure in the remaining coordinate. When the measure spaces are arbitrary or understood, we write $\mathcal{F}_n$ for $\mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be locally compact abelian groups with corresponding dual groups $\hat{\mathcal{X}}_1, \ldots, \hat{\mathcal{X}}_n$. If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are the respective Borel fields of $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and $\mu \in \mathcal{F}_n$, then the Fourier-Stieltjes transform of μ is given by

$$\hat{\mu}(\gamma_1, \ldots, \gamma_n) = \int \bar{\gamma}_1 \otimes \cdots \otimes \bar{\gamma}_n d\mu, \quad (\gamma_1, \ldots, \gamma_n) \in \hat{\mathcal{X}}_1 \times \cdots \times \hat{\mathcal{X}}_n.$$

(The integral above is defined iteratively.) We use the multi-linear Riesz Representation Theorem [B1] to identify $\mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ with the dual space of

$$C_0(\mathcal{X}_1) \hat{\otimes} \cdots \hat{\otimes} C_0(\mathcal{X}_n) \equiv \mathcal{V}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n),$$

and thus extend the action to arbitrary bounded functions by integration. See [B3] for details. It is natural to consider the feasibility of convolution of $\mathcal{F}_n$-measures.

Definition 1. Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be LCA groups with Borel fields $\mathcal{A}_1, \ldots, \mathcal{A}_n$. $\mathcal{F}_n$-measures μ and ν are **convolvable** if $\hat{\mu}\hat{\nu} = \hat{\lambda}$ for some $\lambda \in \mathcal{F}_n$; λ is then denoted by $\mu * \nu$ or $\nu * \mu$. We say $\mu \in \mathcal{F}_n$ is a **convolver** if $\mu * \nu$ exists for all $\nu \in \mathcal{F}_n$.

The case $n = 1$ in Definition 1 is classical; every $\mathcal{F}_1$-measure is a convolver. It is shown in [GS1] that in a two-dimensional setting every $\mathcal{F}_2$-measure is a convolver, while in [GS2] it is shown that there exist non-convolvers in $\mathcal{F}_3$. In general, convolvability is related to *projective boundedness*, a property conveying a Grothendieck-type inequality. In the definition which follows, $\mathcal{L}^\infty(\mathcal{X}_j)$ denotes the space of bounded functions on $\mathcal{X}_j$.

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Definition 2 ([B3]). Let $(\mathcal{X}_1, \mathcal{A}_1), \ldots, (\mathcal{X}_n, \mathcal{A}_n)$ be measurable spaces, and let $\mu \in \mathcal{F}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)$. For $f_j \in \mathcal{L}^\infty(\mathcal{X}_j)$, $j = 1, \ldots, n$, define

$$\phi_\mu(f_1, \ldots, f_n) = \int f_1 \otimes \cdots \otimes f_n d\mu.$$

We say that μ is projectively bounded if

$$\|\mu\|_{pb_n} := \sup\{\|\phi_\mu|_{E_1 \times \cdots \times E_n}\|_{\mathcal{V}_n(E_1, \ldots, E_n)} : E_j \subset Ball(\mathcal{L}^\infty(X_j)), |E_j| < \infty,\ j = 1, \ldots, n\} < \infty;$$

the class of projectively bounded $\mathcal{F}_n$-measures on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ is denoted by $\mathcal{PBF}_n = \mathcal{PBF}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)$.

Let $\mathcal{X}_1, \ldots, \mathcal{X}_n, \mathcal{Y}_1, \ldots, \mathcal{Y}_n$ be locally compact Hausdorff spaces with respective Borel fields $\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}_1, \ldots, \mathcal{B}_n$. For

$$f \in \mathcal{V}_n(\mathcal{X}_1 \times \mathcal{Y}_1, \ldots, \mathcal{X}_n \times \mathcal{Y}_n) \quad \text{and} \quad \mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n),$$

define a function $\eta_{f;\mu}$ on $\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$ by

(1)
$$\eta_{f;\mu}(y_1, \ldots, y_n) = \int f(x_1, y_1, \ldots, x_n, y_n)\mu(dx_1, \ldots, dx_n) \qquad (y_j \in \mathcal{Y}_j).$$

Definition 3. Let $\mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

1. (τ-projective boundedness) Let $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ be locally compact Hausdorff spaces, and define
$$\|\mu\|_{pb_{\tau,n}} := \sup\{\|\eta_{f;\mu}\|_{\mathcal{V}_n(\mathcal{Y}_1, \ldots, \mathcal{Y}_n)} : \|f\|_{\mathcal{V}_n(\mathcal{X}_1 \times \mathcal{Y}_1, \ldots, \mathcal{X}_n \times \mathcal{Y}_n)} \leq 1\}.$$
We say that μ is τ-projectively bounded if $\|\mu\|_{pb_{\tau,n}} < \infty$; the class of τ-projectively bounded $\mathcal{F}_n$-measures on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ is denoted by $\mathcal{PBF}_{\tau,n} = \mathcal{PBF}_{\tau,n}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.
2. (g-projective boundedness) Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be LCA groups. For $\mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and elementary tensors $f = f_1 \otimes \cdots \otimes f_n \in \mathcal{V}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)$, let

(2) $$\Psi_{f;\mu}(y_1, \ldots, y_n) = \int f(x_1 + y_1, \ldots, x_n + y_n)\mu(dx_1, \ldots, dx_n) \qquad (y_j \in \mathcal{X}_j),$$

and define
$$\|\mu\|_{pb_{g,n}} := \sup\{\|\Psi_{f;\mu}\|_{\mathcal{V}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)} : \|f\|_{\mathcal{V}_n} = \prod_j \|f_j\|_\infty \leq 1\}.$$
We say that μ is g-projectively bounded if $\|\mu\|_{pb_{g,n}} < \infty$; the class of g-projectively bounded $\mathcal{F}_n$-measures on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ is denoted by $\mathcal{PBF}_{g,n} = \mathcal{PBF}_{g,n}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

The symbols τ and g in Definition 3 denote, respectively, *topological* and *group-topological* projective boundedness. It is straightforward to check that $\mathcal{PBF}_n$, $\mathcal{PBF}_{g,n}$, and $\mathcal{PBF}_{\tau,n}$ are Banach spaces. To verify that every $\mu \in \mathcal{PBF}_{g,n}$ is a convolver, let $\nu \in \mathcal{F}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ be arbitrary and consider a linear form $\Lambda : \mathcal{V}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n) \to \mathbf{C}$ given by

(3)
$$\Lambda(f) = \int \left(\int f(x_1 + y_1, \ldots, x_n + y_n)\mu(dx_1, \ldots, dx_n) \right) \nu(dy_1, \ldots, dy_n).$$

Then Λ is bounded, and the $\mathcal{F}_n$-measure λ representing this linear action satisfies $\hat{\lambda} = \hat{\mu}\hat{\nu}$. Each $\mu \in \mathcal{PBF}_{g,n}$ can be identified with a bounded linear operator $T_\mu : \mathcal{F}_n \to \mathcal{F}_n$ given by $T_\mu(\nu) = \mu * \nu$. By a routine argument, $\mu \in \mathcal{PBF}_{g,n}$ if and only if the domain of T_μ is $\mathcal{F}_n$, and thus $\mathcal{PBF}_{g,n}$ is precisely the space of convolvers.

For $\mu \in \mathcal{F}_n$, let $\mathcal{D}(\mu) = \{\alpha \in \mathcal{F}_n : \mu * \alpha \text{ exists}\}$, i.e., $\mathcal{D}(\mu)$ is the domain of T_μ. Results in [GS1] and [GS2] state that in a two-dimensional setting $\mathcal{D}(\mu) = \mathcal{F}_2$ for all $\mu \in \mathcal{F}_2$, *and* there exist $\mu \in \mathcal{F}_3$ with $\mathcal{D}(\mu) \subsetneqq \mathcal{F}_3$. Thus it is natural to ask: *for* $\mu \in \mathcal{F}_2(\sigma(\mathcal{A}_1 \times \mathcal{A}_2), \mathcal{A}_3)$, *is* $\mathcal{D}(\mu) = \mathcal{F}_3(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$*?* We show that the answer is *no.*

Let $\mathcal{S}(\mathcal{A}_j)$ be the space of $\mathcal{A}_j$-simple functions on $\mathcal{X}_j$, equipped with the uniform norm. $\mathcal{V}_n(\mathcal{A}_1, \dots, \mathcal{A}_n)$ will denote the completion of $\mathcal{S}(\mathcal{A}_1) \otimes \cdots \otimes \mathcal{S}(\mathcal{A}_n)$ with respect to the projective tensor norm.

Proposition 4. $\mathcal{PBF}_n \subset \mathcal{PBF}_{\tau,n} \subset \mathcal{PBF}_{g,n}$.

Proof. The second inclusion is immediate from the definitions. Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be compact spaces. A theorem of Saeki [S] states that

$$C(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n) \cap \mathcal{V}_n(\mathcal{A}_1, \dots, \mathcal{A}_n) = \mathcal{V}_n(\mathcal{X}_1, \dots, \mathcal{X}_n). \tag{4}$$

We show that

$$\|\eta_{f;\mu}\|_{\mathcal{V}_n(\mathcal{Y}_1,\dots,\mathcal{Y}_n)} \le \|f\|_{\mathcal{V}_n} \|\mu\|_{pb_n}, \;\; f \in \mathcal{V}_n(\mathcal{X}_1 \times \mathcal{Y}_1, \dots, \mathcal{X}_n \times \mathcal{Y}_n). \tag{5}$$

(See (1) for the definition of $\eta_{f;\mu}$.) Let $g = g_1 \otimes \cdots \otimes g_n$ be an elementary tensor in $C_c(\mathcal{X}_1 \times \mathcal{Y}_1, \dots, \mathcal{X}_n \times \mathcal{Y}_n)$. (The subscript c denotes compact support.) Then $\eta_{g;\mu} \in C_c(\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n)$. For locally compact spaces X and Y, $\mathcal{V}_2(X, Y)$ is uniformly dense in $C_0(X \times Y)$, and thus for $k = 1, \dots, n$ there exist sequences $\{\rho_{jk}\} \subset \mathcal{S}(\mathcal{A}_k) \otimes \mathcal{S}(\mathcal{B}_k)$ with $\lim_{j\to\infty} \rho_{jk} = g_k$ in the uniform norm. Let $\theta_j = \rho_{j1} \otimes \cdots \otimes \rho_{jn}$. Then $\lim_j \eta_{\theta_j;\mu} = \eta_{g;\mu}$ (uniform limit), and $\{\eta_{\theta_j;\mu}\}$ is a $\mathcal{V}_n(\mathcal{B}_1, \dots, \mathcal{B}_n)$-Cauchy sequence satisfying

$$\|\eta_{\theta_j;\mu}\|_{\mathcal{V}_n(\mathcal{B}_1,\dots,\mathcal{B}_n)} \le \|\rho_{j1}\|_\infty \cdots \|\rho_{jn}\|_\infty \|\mu\|_{pb_n}.$$

By (4), $\eta_{g;\mu} \in \mathcal{V}_n(\mathcal{Y}_1, \dots, \mathcal{Y}_n)$. Let $\epsilon > 0$, and choose $f \in \mathcal{V}_n(\mathcal{X}_1 \times \mathcal{Y}_1, \dots, \mathcal{X}_n \times \mathcal{Y}_n)$. Without loss of generality, we can take f to be compactly supported. We represent

$$f = \sum_k g_k,$$

where $g_k = g_{1k} \otimes \cdots \otimes g_{nk}$, and

$$\|f\|_{\mathcal{V}_n} \le (1 + \epsilon) \sum_k \|g_{1k}\|_\infty \cdots \|g_{nk}\|_\infty,$$

with each $g_k \in C_c(\mathcal{X}_1 \times \mathcal{Y}_1) \otimes \cdots \otimes C_c(\mathcal{X}_n \times \mathcal{Y}_n)$. Then $\eta_{f;\mu} = \sum_k \eta_{g_k;\mu}$, and (5) follows. ∎

We do not know which of the inclusions in Proposition 4 are proper. ($\mathcal{PBF}_n$ is properly contained in $\mathcal{PBF}_{g,n}$, as shown by the 'fractional' forms in [B3].)

We now show that in a three-dimensional setting, an $\mathcal{F}_2$-measure need not be a convolver.

Lemma 5. *Let* $\mathcal{X}_1$, $\mathcal{X}_2$, *and* $\mathcal{X}_3$ *be infinite locally compact abelian groups with respective Borel fields* $\mathcal{A}_1, \mathcal{A}_2$, *and* $\mathcal{A}_3$. *For every* $K > 0$ *there exists a discrete measure* μ *on* $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ *satisfying* $\|\mu\|_{\mathcal{F}_2(\sigma(\mathcal{A}_1\times\mathcal{A}_2),\mathcal{A}_3)} \le 1$ *and* $\|\mu\|_{pb_{g,3}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)} \ge K$.

Proof. Let $\mathbf{Z}_n$ be the group of integers under addition modulo n, and let $\hat{\mathbf{Z}}_n$ be its dual. Let $\mu_n : C(\hat{\mathbf{Z}}_n \times \hat{\mathbf{Z}}_n) \times C(\mathbf{Z}_n) \to \mathbf{C}$ be given by

$$\mu_n(f,h) = \sum_{m \in \hat{\mathbf{Z}}_n} \frac{f(m,m)}{\sqrt{n}} \hat{h}(m).$$

We show that $\|\mu_n\|_{\mathcal{F}_2(\hat{Z}_n \times \hat{Z}_n, Z_n)} \leq 1$ and $\|\hat{\mu}_n\|_{\mathcal{V}_3(Z_n, Z_n, \hat{Z}_n)} \to \infty$ as $n \to \infty$. (For convenience, we write $\mathcal{F}_2(\hat{Z}_n \times \hat{Z}_n, Z_n)$ for $\mathcal{F}_2(2^{\hat{Z}_n \times \hat{Z}_n}, 2^{Z_n})$.) Clearly, $\|\mu_n\|_{\mathcal{F}_2} \leq 1$ (by Cauchy-Schwarz). μ_n is also an element of $\mathcal{F}_3(\hat{\mathbf{Z}}_n, \hat{\mathbf{Z}}_n, \mathbf{Z}_n)$:

$$\mu_n(f,g,h) = \sum_{m \in \hat{\mathbf{Z}}_n} \frac{f(m)g(m)}{\sqrt{n}} \hat{h}(m).$$

Then

$$\hat{\mu}_n(j,k,l) = \sum_{m \in \hat{\mathbf{Z}}_n} \frac{e^{-\frac{2\pi i j m}{n}} e^{-\frac{2\pi i k m}{n}}}{n\sqrt{n}} \sum_{s \in \mathbf{Z}_n} e^{\frac{-2\pi i (l+m)s}{n}}$$

$$= \frac{1}{\sqrt{n}} e^{2\pi i j l/n} e^{2\pi i k l/n}.$$

Define

$$\Phi_n(j,k,l) = \frac{1}{n^2} e^{-2\pi i j l/n} e^{-2\pi i k l/n}.$$

Let $\mathbf{s}, \mathbf{t}$, and $\mathbf{u}$ be elements in the unit ball of $\ell^\infty(\mathbf{Z}_n)$. Then

$$|\sum_{j,k,l} \Phi_n(j,k,l)\mathbf{s}(j)\mathbf{t}(k)\mathbf{u}(l)|$$

$$= \frac{1}{n^2} |\sum_l \mathbf{u}(l) \sum_j \mathbf{s}(j) e^{-2\pi i j l/n} \sum_k \mathbf{t}(k) e^{-2\pi i k l/n}|$$

$$= |\sum_l \mathbf{u}(l)\hat{\mathbf{s}}(l)\hat{\mathbf{t}}(l)| \leq \|\mathbf{u}\|_\infty \|\hat{\mathbf{s}}\|_2 \|\hat{\mathbf{t}}\|_2 \leq \|\mathbf{s}\|_\infty \|\mathbf{t}\|_\infty \leq 1.$$

Thus, $\|\Phi_n\|_{\mathcal{F}_3} \leq 1$. However,

$$|\sum_{j,k,l} \hat{\mu}_n(j,k,l)\Phi_n(j,k,l)| = \sqrt{n}.$$

Therefore, by the duality $(\mathcal{V}_3)^* = \mathcal{F}_3$,

$$\sqrt{n} \leq \|\hat{\mu}_n\|_{\mathcal{V}_3} \leq \|\mu_n\|_{pb_3}.$$

Let $[N] = \{1, \ldots, N\}$. Given $K > 0$, there exist N and $\mu \in \mathcal{F}_2([N]^2, [N])$ with $\|\mu\|_{\mathcal{F}_2} \leq 1$ and $\|\mu\|_{pb_3} > K$. If $\mu = \{\mu_{xyz} : (x,y,z) \in [N]^3\}$, there exist arrays a, b, c in the unit ball of $\ell^\infty([N]^2)$ such that

$$\|\eta_{a,b,c;\mu}\|_{\mathcal{V}_3([N],[N],[N])} > K,$$

where

$$\eta_{a,b,c;\mu}(i,j,k) = \sum_{(x,y,z) \in [N]^3} \mu_{xyz} a_{xi} b_{yj} c_{zk}, \quad (i,j,k) \in [N]^3.$$

Let F_m and G_m be disjoint mutually independent subsets of $\mathcal{X}_m$ of cardinality N given by

$$F_m = \{s_{jm} : j \in [N]\}, \;\; G_m = \{t_{jm} : j \in [N]\}, \quad m = 1, 2, 3.$$

(Disjoint subsets A and B of an abelian group are *mutually independent* if given elements (a_1, b_1) and (a_2, b_2) of $A \times B$, the relation $a_1 + b_1 = a_2 + b_2$ implies that $a_1 = a_2$ and $b_1 = b_2$.) Define a measure $\tilde{\mu}$ on $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ by

$$\tilde{\mu} = \sum_{(x,y,z)\in[N]^3} \mu_{xyz} \delta_{s_{x1}} \otimes \delta_{s_{y2}} \otimes \delta_{s_{z3}},$$

and observe that $\|\tilde{\mu}\|_{\mathcal{F}_2(\sigma(\mathcal{A}_1\times\mathcal{A}_2),\mathcal{A}_3)} \leq 1$. Therefore, by the independence of F_m and G_m, we can find $f \in C_0(\mathcal{X}_1)$, $\;g \in C_0(\mathcal{X}_2)$, and $h \in C_0(\mathcal{X}_3)$ with

$$f(s_{i1} + t_{j1}) = a_{ij}, \quad g(s_{i2} + t_{j2}) = b_{ij}, \quad h(s_{i3} + t_{j3}) = c_{ij},$$

so that

$$\eta_{a,b,c;\mu}(i, j, k) = \Psi_{f\otimes g\otimes h;\tilde{\mu}}(t_{i1}, t_{j2}, t_{k3}), \;\; (i, j, k) \in [N]^3.$$

(Refer to (1) and (2) for definitions of η and Ψ.) Thus $\|\Psi_{f\otimes g\otimes h;\mu}\|_{\mathcal{V}_3} \geq K$, and hence $\|\tilde{\mu}\|_{pb_{g,n}} \geq K$. ☐

Corollary 6. *If the underlying σ-algebras $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$ are infinite, there exists $\mu \in \mathcal{F}_2(\sigma(\mathcal{A}_1 \times \mathcal{A}_2), \mathcal{A}_3)$ which is not a convolver in $\mathcal{F}_3$.*

Two further questions

1. It is shown in [B3] that scalar measures are projectively bounded. Hence, by Proposition 4, all scalar measures are convolvers. An $\mathcal{F}_3$-measure μ is an $\mathcal{F}_{\frac{3}{2}}$-measure if, when any one coordinate is fixed, μ extends to a scalar measure in the remaining two coordinates. (See [B2] for details.) We have the proper containments $\mathcal{F}_1 \subsetneq \mathcal{F}_{\frac{3}{2}} \subsetneq \mathcal{F}_2$. We do not know whether all $\mathcal{F}_{\frac{3}{2}}$-measures are convolvers.

2. Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be LCA groups. It is shown in [ZS] that the space of completely bounded n-linear forms on $C_0(\mathcal{X}_1) \times \cdots \times C_0(\mathcal{X}_n)$ has a natural Banach *-algebra structure extending that of $\mathcal{F}_1$ on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. We do not know whether all completely bounded forms are convolvers, or if all convolvers are completely bounded.

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