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WEAK TYPE ESTIMATES FOR CONE MULTIPLIERS ON H^p SPACES, $p < 1$

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Abstract. We consider operators T^δ associated with the Fourier multipliers

$$\left(1 - \frac{|\xi'|^2}{\xi_{n+1}^2}\right)_+^\delta, \qquad (\xi', \xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R},$$

and show that T^δ is of weak type (p,p) on $H^p(\mathbb{R}^{n+1})$, $0 < p < 1$, for the critical value $\delta = n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$.

1. Introduction

We consider the family of Fourier multipliers

$$m^\delta(\xi', \xi_{n+1}) = \left(1 - \frac{|\xi'|^2}{\xi_{n+1}^2}\right)_+^\delta, \qquad (\xi', \xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R},$$

with $m^\delta(\xi', \xi_{n+1}) = 0$ when $|\xi'| > \xi_{n+1}$. Let $\widehat{f}$ be the Fourier transform of a Schwartz function f on $\mathbb{R}^n \times \mathbb{R}$. Then define convolution operators T^δ by

$$\widehat{T^\delta f}(\xi', \xi_{n+1}) = m^\delta(\xi', \xi_{n+1})\widehat{f}(\xi', \xi_{n+1}).$$

It is conjectured that T^δ is bounded on L^p and it remains open for any $n \geq 2$. No optimal L^p-bounds are known for $p > 1$. For partial results, see G. Mockenhaupt [3] and J. Bourgain [1]. See also Mockenhaupt, Seeger and Sogge [4] for the related results on the wave equation. When $\delta > (n-1)/2$, it is not hard to show that T^δ is of weak type $(1,1)$ by using Calderón-Zygmund theory.

The purpose of this paper is to prove a sharp endpoint result on $H^p(\mathbb{R}^{n+1})$, $p < 1$. This estimate implies the known result due to Stein, Taibleson and Weiss [7] that the Bochner-Riesz means of the critical index $\delta_p = n(1/p - 1/2) - 1/2$ is of weak type (p,p) for functions in $H^p(\mathbb{R}^n)$ (see the Appendix). Here H^p is the standard real Hardy space as defined in [6] by E. Stein.

We prove here

Theorem 1. *Suppose* $0 < p < 1$ *and* $\delta = n(1/p - 1/2) - 1/2$. *Then* T^δ *maps* $H^p(\mathbb{R}^{n+1})$ *boundedly into weak-*$L^p(\mathbb{R}^{n+1})$*; i.e., there exists a constant* $C = C(n,p)$

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such that for all $f \in H^p(\mathbb{R}^{n+1})$

$$|\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |T^\delta f(x,t)| > \alpha\}| \le C\left(\frac{||f||_{H^p(\mathbb{R}^{n+1})}}{\alpha}\right)^p$$

for all $\alpha > 0$, *where* $|E|$ *denotes the Lebesgue measure of the set* $E \subset \mathbb{R}^{n+1}$.

It is not known whether $T^{\frac{n-1}{2}}$ is of weak type $(1,1)$ or just of weak type $(1,1)$ on functions in H^1.

With respect to the notation, we use C to denote a positive constant, whose value may be different for each occurrence.

2. Kernel estimates

Let $\varphi, \psi \in C_0^\infty(\mathbb{R})$ be supported in $(1/2, 2)$ such that $\sum_{k\ge1} \varphi(2^k s) = 1$ and $\sum_{l=-\infty}^{\infty} \psi(2^{-l}t) = 1$ for $0 < s < 1$, $t > 0$. We now fix k and l. We shall need pointwise estimates for the kernels of

$$T_{k,l}^\delta f(x,t) = (2\pi)^{-(n+1)} \int_{\mathbb{R}}\int_{\mathbb{R}^n} G_{k,l}(x-y, t-s) f(y,s)\, dy\, ds$$

where

$$G_{k,l}(x,t) = \int_{\mathbb{R}}\int_{\mathbb{R}^n} \varphi\left(2^k\left(1 - \frac{|\xi'|^2}{\xi_{n+1}^2}\right)\right)\left(1 - \frac{|\xi'|^2}{\xi_{n+1}^2}\right)_+^\delta \psi(2^{-l}\xi_{n+1}) \times e^{i\langle x,\xi'\rangle + it\xi_{n+1}}\, d\xi'\, d\xi_{n+1}. \tag{2.1}$$

For each k and l, the kernel $G_{k,l}$ has the property

$$G_{k,l}(\cdot,\cdot) = 2^{l(n+1)}\,\{G_{k,0}(2^l\cdot, 2^l\cdot)\}, \tag{2.2}$$

and we write $\sum_{k\ge1} G_{k,l} = G_l = 2^{l(n+1)}\, G_0(2^l\cdot, 2^l\cdot)$ and $\sum_{k\ge1} T_{k,l}^\delta = T_l^\delta$.

Lemma 1. *Suppose* $2^l|x| \le 2$ *for fixed* l. *Then for each* k *there is an estimate as follows: for every* N

$$|G_{k,l}(x,t)| \le C\, 2^{l(n+1)}\, 2^{-k(\delta+1)}\, \min\{1, (2^{-k}\, 2^l|t|)^{-N}\}. \tag{2.3}$$

Thus,

$$|G_l(x,t)| \le C\, 2^{l(n+1)} \frac{1}{(1+2^l|t|)^{\delta+1}}. \tag{2.4}$$

Proof. In view of (2.1) and (2.2), it suffices to show (2.3) and (2.4) for $l = 0$. We integrate by parts N times with respect to ξ_{n+1} in (2.1). In view of the support property of φ, (2.3) follows. Using (2.3) gives

$$\left\{C \sum_{|t|\le 2^k} 2^{-k(\delta+1)} + C \sum_{|t|>2^k} 2^{-k(\delta+1-N)}\, |t|^{-N}\right\},$$

and thus (2.4) is established. □

Now we estimate the kernel for the case $2^l|x| > 2$. We let $|x| = r$ and define $G_{k,l}(x,t) = K_{k,l}(|x|,t)$. By Bochner's formula [8] and change of variables, we have

$$\begin{aligned} K_{k,l}(r,t) \;=\; & 2^{l(n+1)}\,(2^l r)^{-(n-2)/2} \int_{\mathbb{R}}\int_0^1 J_{\frac{n-2}{2}}(\rho(2^l r)\xi_{n+1})\, \varphi(2^k(1-\rho^2)) \\ & \times (1-\rho^2)^\delta \psi(\xi_{n+1})\xi_{n+1}^{(n+2)/2} \rho^{n/2} e^{i(2^l t)\xi_{n+1}}\, d\rho\, d\xi_{n+1}. \end{aligned} \tag{2.5}$$

Here J_μ is the Bessel function of order $\mu > -\frac{1}{2}$ defined by

$$J_\mu(t) = A_\mu t^\mu \int_{-1}^{1} e^{it\sigma}(1-\sigma^2)^{\mu-\frac{1}{2}}\, d\sigma \tag{2.6}$$

where $A_\mu = [2^\mu \Gamma(2\mu+1)\Gamma(\frac{1}{2})]^{-1}$.

In order to estimate the kernel (2.5), we need some properties of Bessel functions [8]

$$\frac{d}{dt}\{t^{-\mu}J_\mu(t)\} = -t\{t^{-(\mu+1)}J_{\mu+1}(t)\}. \tag{2.7}$$

For the following lemma, we use dyadic decompositions of Bessel functions (2.6) following the article by Müller and Seeger [5].

Lemma 2. *Suppose that $2^l r > 2$ for fixed l. Then for each k there is an estimate as follows:*

$$\begin{aligned} |K_{k,l}(r,t)| \quad &\leq \quad C\, 2^{l(n+1)}\, 2^{-k(\delta+1)}\, (2^l r)^{-(n-1)/2} \, \min\{1, (2^{-k}\, 2^l r)^{-N_1}\} \\ &\qquad \times \left\{ \frac{1}{(1+2^l|t+r|)^N} + \frac{1}{(1+2^l|t-r|)^N} \right\}. \end{aligned} \tag{2.8}$$

Moreover,

(2.9)

$$|K_l(r,t)| \leq C\, 2^{l(n+1)}\, (1+2^l r)^{-(n+1+2\delta)/2} \left\{ \frac{1}{(1+2^l|t+r|)^N} + \frac{1}{(1+2^l|t-r|)^N} \right\}.$$

Proof. Let $\eta \in C_0^\infty(\mathbb{R})$ be supported in $(-1/2, 2)$ and equal to 1 in $(-1/4, 1/4)$. Define $m = 0, 1, 2, \cdots$ and

$$\eta_{mk}(\sigma,\nu) = \begin{cases} \eta(2^{-k}\nu(1-\sigma^2)) & \text{if} \quad m = 0, \\ \eta(2^{-k-m}\nu(1-\sigma^2)) - \eta(2^{-k-m+1}\nu(1-\sigma^2)) & \text{if} \quad m > 0 \end{cases}$$

and set

$$J_{\mu,k}^m(\rho\nu) = A_\mu(\rho\nu)^\mu \int_{-1}^{1} e^{i(\rho\nu)\sigma}\, (1-\sigma^2)^{\mu-1/2}\eta_{mk}(\sigma,\nu)\, d\sigma.$$

Let $M > N + N_1 + (n-1)/2$ and set

$$\phi_{mk\nu}(\sigma) = \begin{cases} (1-\sigma^2)^{(n-3)/2}\, \eta_{mk}(\sigma,\nu) & \text{if} \quad m = 0\ , \\ \left(\frac{1}{i\rho\nu}\right)^M \left(\frac{d}{d\sigma}\right)^M [\eta_{mk}(\sigma,\nu)(1-\sigma^2)^{(n-3)/2}] & \text{if} \quad m > 0. \end{cases}$$

Then

$$J_{\frac{n-2}{2},k}^m(\rho\nu) = A_{\frac{n-2}{2}}(\rho\nu)^{\frac{n-2}{2}} \int_{-1}^{1} e^{i(\rho\nu)\sigma}\phi_{mk\nu}(\sigma)\, d\sigma \tag{2.10}$$

by integration by parts if $m > 0$.

Fix k and set $l = 0, \nu = r\xi_{n+1}$. We may decompose the kernel (2.5) as

$$K_{k,0} = \sum_{m=0}^{\infty} K_{k,0}^m$$

where

$$K^m_{k,0}(r,t) = r^{-(n-2)/2} \int_{\mathbb{R}} \int_0^1 J^m_{\frac{n-2}{2},k}(\rho\nu)\, \varphi(2^k(1-\rho^2)) \times (1-\rho^2)^\delta \psi(\xi_{n+1})\, \xi_{n+1}^{(n+2)/2} \rho^{n/2} e^{it\xi_{n+1}}\, d\rho\, d\xi_{n+1}.$$

Formula (2.10) and straightforward computation imply that

$$K^m_{k,0}(r,t) = A_{\frac{n-2}{2}} \int_{-1}^1 \phi_{mk\nu}(\sigma) \int_{\mathbb{R}} \int_0^1 \varphi(2^k(1-\rho^2)) \times (1-\rho^2)^\delta \psi(\xi_{n+1}) \xi^n_{n+1} \rho^{n-1}\, e^{i(t+\rho r\sigma)\xi_{n+1}}\, d\rho\, d\xi_{n+1}\, d\sigma. \tag{2.11}$$

We integrate by parts with respect to ρ and ξ_{n+1} in (2.11) and by Fubini's theorem

$$|K^m_{k,0}(r,t)| \leq C\, 2^{-k(n-1)/2} \int_{1/2}^2 \int_{-1}^1 \int_0^1 |\phi_{mk\nu}(\sigma)|\, (1+|\sigma\nu|)^{-N_1} \times (1+|t+\rho r\sigma|)^{-N} \left| \left(\frac{\partial}{\partial\rho}\right)^{N_1} \varphi(2^k(1-\rho^2))(1-\rho^2)^\delta \rho^{(n-1)/2} \right| \times \left| \left(\frac{\partial}{\partial\xi_{n+1}}\right)^N \psi(\xi_{n+1}) \xi^n_{n+1} \right| d\rho\, d\sigma\, d\xi_{n+1}. \tag{2.12}$$

Next note the size estimate

$$|\phi_{mk\nu}(\sigma)| \leq C\, 2^{-mM}\, (2^{m+k}\nu^{-1})^{(n-3)/2}. \tag{2.13}$$

Moreover, $\phi_{mk\nu}$ vanishes unless either $1-\sigma^2 \approx 2^{m+k}\nu^{-1}$ for $m > 0$, or $1-\sigma^2 \leq 2^k\nu^{-1}$ for $m = 0$. Hence if σ is in the support of $\phi_{mk\nu}$, then either $|\nu - \nu\sigma| \leq 2^{m+k}$ or $|\nu + \nu\sigma| \leq 2^{m+k}$. Then using the estimates (2.13), the integrand of (2.12) is bounded by

$$\begin{aligned} & C\, 2^{-k\delta} |\phi_{mk\nu}(\sigma)| \frac{1}{(1+2^{-k}|\sigma\nu|)^{N_1}} \frac{1}{(1+|t+\rho r\sigma|)^N} \\ \leq\ & C\, 2^{k\{(n-3)/2-\delta\}}\, 2^{m((n-3)/2+N+N_1-M)}\, \nu^{-(n-3)/2}\, \xi^{-N}_{n+1} \\ & \times \frac{1}{(1+2^{-k}|\nu|)^{N_1}} \left\{ \frac{1}{(1+|t+\rho r|)^N} + \frac{1}{(1+|t-\rho r|)^N} \right\}. \end{aligned}$$

If we integrate over the support of $\varphi(2^k(1-\rho^2)) \otimes \phi_{mk\nu} \otimes \psi$ for $m \geq 0$ in (2.12), we gain an additional factor of $C\, 2^m r^{-1}$. Since $M > N + N_1 + (n-1)/2$, we may sum over m and the desired estimates (2.8) follow from (2.12). Hence we obtain

$$(1+||t|-r|)^{-N} \left\{ C \sum_{r \leq 2^k} 2^{-k(\delta+1)}\, r^{-(n-1)/2} + C \sum_{r > 2^k} 2^{-k(\delta+1-N_1)}\, r^{-(n-1)/2-N_1} \right\},$$

and thus (2.9) is established for $l = 0$. For the case $l \neq 0$, we use (2.2). □

In Section 4 we will need estimates for the derivatives of the kernels. When $|x| \leq 2$, straightforward computations in (2.1) give us Lemma 3(a). If $|x| = r > 2$, we use (2.7) to show Lemma 3(b).

Lemma 3. (a) *Suppose that* $|x| \le 2$. *Then for every* N,

$$|G_{k,0}^{(\gamma)}(x,t)| \le C\, 2^{-k(\delta+1)} \min\{1, (2^{-k}|t|)^{-N}\}.$$

Moreover,

$$\begin{aligned} |G_0^{(\gamma)}(x,t)| &\le C \sum_{|t|\le 2^k} 2^{-k(\delta+1)} + C \sum_{|t|>2^k} 2^{-k(\delta+1-N)}|t|^{-N} \\ &\le C \frac{1}{(1+|t|)^{\delta+1}} \quad \text{for } \gamma \in \mathbb{N}^{n+1} \text{ and } |\gamma| = 1,2,3,\cdots. \end{aligned}$$

(b) *Suppose that* $r > 2$. *Then*

$$|K_{k,0}^{(\gamma)}(r,t)| \le C\, 2^{-k(\delta+1)}\, r^{-(n-1)/2} \min\{1, (2^{-k}r)^{-N_1}\}(1+||t|-r|)^{-N}.$$

Moreover,

$$\begin{aligned} |K_0^{(\gamma)}(r,t)| &\le (1+||t|-r|)^{-N} \Big\{ C \sum_{r\le 2^k} 2^{-k(\delta+1)}\, r^{-(n-1)/2} \\ &\qquad + C \sum_{r>2^k} 2^{-k(\delta+1-N_1)}\, r^{-(n-1)/2-N_1} \Big\} \\ &\le C\,(1+r)^{-(n+1+2\delta)/2} \Big\{ \frac{1}{(1+|t+r|)^N} + \frac{1}{(1+|t-r|)^N} \Big\} \end{aligned}$$

for $\gamma \in \mathbb{N}^2$ *and* $|\gamma| = 1,2,3,\cdots$.

3. The atomic decomposition of H^p and preliminary lemmas

Definition 1. Let $0 < p \le 1$ and d be an integer that satisfies $d \ge (n+1)(1/p-1)$. Let Q be a cube in $\mathbb{R}^{n+1}$. We say that a is a (p,d)-atom associated with Q if a is supported on $Q \subset \mathbb{R}^{n+1}$ and satisfies

$$\text{(3.1)} \qquad \begin{aligned} &\text{(i)} \quad |a(x)| \le |Q|^{-1/p} \qquad \text{almost everywhere} \\ &\text{(ii)} \quad \int_{\mathbb{R}^{n+1}} a(x)x^\beta\, dx = 0 \end{aligned}$$

where $\beta = (\beta_1, \beta_2, \cdots, \beta_{n+1})$ is an $(n+1)$-tuple of non-negative integers satisfying $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_{n+1} \le d$, and $x^\beta = x^{\beta_1} x^{\beta_2} \cdots x^{\beta_{n+1}}$.

If $\{a_i\}$ is a collection of (p,d)-atoms and $\{\lambda_i\}$ is a sequence of complex numbers with $\sum_i |\lambda_i|^p < \infty$, then the series

$$f = \sum_{i=1}^{\infty} \lambda_i a_i$$

converges in the sense of distributions, and its sum belongs to H^p with $||f||_{H^p} \le C\,(\sum_i |\lambda_i|^p)^{1/p}$ and the converse inequality also holds (see [6]).

The following lemma is due to Stein, Taibleson and Weiss [7]. It is a version of the triangle inequality on $L^{p,\infty}$, $0 < p < 1$.

Lemma 4. *Suppose* $0 < p < 1$ *and* $\{f_i\}$ *is a sequence of mesaurable functions such that*

$$|\{x : |f_i(x)| > \alpha > 0\}| \le \alpha^{-p} \tag{3.2}$$

for $i = 1, 2, 3, \cdots$. *If* $\sum_{i=1}^{\infty} |\lambda_i|^p \le 1$, *then*

$$\left|\left\{x : \left|\sum_{i=1}^{\infty} \lambda_i f_i(x)\right| > \alpha\right\}\right| \le \frac{2-p}{1-p}\alpha^{-p}. \tag{3.3}$$

Proof. See ([7]). □

Lemma 5. *For given* $\beta > 0$, *assume that* $0 < p < 1$. *Suppose that* $\{g_l\}$ *is a sequence of measurable functions such that*

$$|\{x : |g_l(x)| > \alpha\}| \le A^p\, 2^{-\beta l p}\, \alpha^{-p} \tag{3.4}$$

for $l \ge 0$ *and all* $\alpha > 0$. *Then*

$$\left|\left\{x : \sum_{l\ge 0} |g_l(x)| > \alpha\right\}\right| \le C\, \alpha^{-p}.$$

Proof. From (3.4), we have

$$\left|\left\{x : |g_l(x)| \cdot \left(\frac{2^{\beta l}}{A}\right) > \alpha\right\}\right| \le \quad A^p\, 2^{-\beta l p}(A\, 2^{-\beta l}\alpha)^{-p} \quad = \quad \alpha^{-p}.$$

Moreover, $\sum_{l\ge 0} A^p\, 2^{-\beta l p} = A_\beta^p\, A^p$ where $A_\beta^p = \dfrac{1}{1 - 2^{-\beta p}}$.

By Lemma 4, we then obtain

$$\begin{aligned} \left|\left\{x : \sum_{l\ge 0} |g_l(x)| > \alpha\right\}\right| \quad &\le \quad \left|\left\{x : \sum_{l\ge 0} \frac{A 2^{-\beta l}}{A_\beta\, A} \frac{|g_l(x)|}{A\, 2^{-\beta l}} > \frac{\alpha}{A_\beta\, A}\right\}\right| \\ &\le \quad \left(\frac{2-p}{1-p}\right)\left(\frac{\alpha}{A_\beta\, A}\right)^{-p} = \quad C\, \alpha^{-p}. \end{aligned}$$

□

The following technical estimates will be used in Section 4.

Lemma 6. *Suppose* $0 < p < 1$ *and* $\delta = n(1/p - 1/2) - 1/2$. *Suppose*

$$\begin{aligned} &\iint_{\{|x|>2^{1-l},\, |t|>2^{1-l}:\ 2^{la}\, |x|^{-n/p}\, \chi_{\{||t|-|x||\le 2^{-l}\}} > \alpha/16C\}} dx\, dt \\ &+ \iint_{\{|x|>2^{1-l},\, |t|>2^{1-l}:\ 2^{lb}\, |x|^{-n/p}\, |t|^{-N} > \alpha/16C\}} dx\, dt \\ &+ \iint_{\{|x|>2^{1-l},\, |t|\le 2^{1-l}:\ 2^{lb}\, |x|^{-n/p-N} > \alpha/16C\}} dx\, dt \\ &+ \iint_{\{|x|\le 2^{1-l},\, |t|>2^{1-l}:\ 2^{lc}\, |t|^{-(\delta+1)} > \alpha/16C\}} dx\, dt \\ \le \quad & C\, 2^{ld}\, \alpha^{-p}. \end{aligned}$$

Then

(i) *if* $a = n+1-n/p$, $b = n+1-n/p-N$ *and* $c = n-\delta$, *then* $d = (n+1)(p-1)$,

(ii) *if* $a = n+N+2-n/p$, $b = n+2-n/p$ *and* $c = n+N+1-\delta$, *then* $d = (n+N+2)p-(n+1)$.

Proof. Applying Fubini's theorem to the first integral and Chebyshev's inequality to the third and last integrals yields the desired estimates. □

4. Weak type estimates

We will show that $T^\delta f$ satisfies the uniform weak type estimates (3.2) when f is a (p,N)-atom ($N \geq (n+1)(1/p-1)$), and prove Theorem 1.

Proposition 1. *Suppose f is a (p,N)-atom ($N \geq (n+1)(1/p-1)$) on $\mathbb{R}^{n+1}$ and $\delta = n(1/p-1/2)-1/2$. Then there exists a constant $C = C(n,p)$ such that*

$$|\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |T^\delta f(x,t)| > \alpha\}| \leq C\,\alpha^{-p} \tag{4.1}$$

for all $\alpha > 0$.

Proof. Since T^δ is translation invariant, we can assume that f is supported in a cube Q of diameter 2^R centered at the origin. We observe

$$\begin{aligned} & |\{(x,t) : |T^\delta f(x,t)| > \alpha\}| \\ \leq \; & |\{(x,t) \in Q^* : |T^\delta f(x,t)| > \alpha/2\}| \\ & + |\{(x,t) \in (Q^*)^c : |T^\delta f(x,t)| > \alpha/2\}| \end{aligned}$$

where Q^* is the cube concentric with Q and with sides of twice the length, and we will show that each term is bounded by $C\,\alpha^{-p}$.

Consider $(x,t) \in Q^*$. Suppose $0 < p < 2$ and $p/2 + 1/q = 1$. By Hölder's inequality and the Plancherel theorem, we have

$$\begin{aligned} \iint_{Q^*} |T^\delta f(x,t)|^p\,dx\,dt \;&\leq\; \left(\iint_{\mathbb{R}^{n+1}} |T^\delta f|^2 dx\,dt\right)^{p/2} |Q^*|^{1/q} \\ &\leq\; C. \end{aligned}$$

Hence for all $\alpha > 0$,

$$|\{(x,t) \in Q^* : |T^\delta f(x,t)| > \alpha/2\}| \leq C\,\alpha^{-p}. \tag{4.2}$$

Next, to establish the following estimate

$$|\{(x,t) \in (Q^*)^c : |T^\delta f(x,t)| > \alpha/2\}| \leq C\,\alpha^{-p} \tag{4.3}$$

for all $\alpha > 0$, we first assume that f is supported in the cube I of diameter 1 centered at the origin. We now consider the case $(x,t) \in (I^*)^c$ and claim that

$$\begin{aligned} |T_l^\delta f(x,t)| \;\leq\; C\Big\{ & 2^{la}\,|x|^{-n/p} \chi_{\{|x|>2^{1-l},\,|t|>2^{1-l},\,||t|-|x||\leq 2^{-l}\}} \\ & + 2^{lb}\,|x|^{-n/p}\,|t|^{-N} \chi_{\{|x|>2^{1-l},\,|t|>2^{1-l}\}} \\ & + 2^{lb}\,|x|^{-n/p-N} \chi_{\{|x|>2^{1-l},\,|t|\leq 2^{1-l}\}} \\ & + 2^{lc}\,|t|^{-(\delta+1)} \chi_{\{|x|\leq 2^{1-l},\,|t|>2^{1-l}\}} \Big\} \end{aligned} \tag{4.4}$$

where

(i) $a = n+1-n/p$, $b = n+1-n/p-N$ and $c = n-\delta$ for $l \geq 0$,
(ii) $a = n+N+2-n/p$, $b = n+2-n/p$ and $c = n+N+1-\delta$ for $l < 0$.

Fix $l \geq 0$. Since f is supported in the cube I of diameter 1, by (2.2) we have

$$|T_l^\delta f(x,t)| \leq 2^{l(n+1)} \iint_{I^*} |f(y,s)|\,|K_0(2^l|x-y|, 2^l(t-s))| dy\,ds.$$

Consider the case $2^l|x| > 2$, $2^l|t| > 2$ and $2^l||t| - |x|| \le 1$. Then by Lemma 2, we have

$$|T_l^\delta f(x,t)| \le C\, 2^{l(n+1-n/p)}\, |x|^{-n/p}\, \chi_{\{||t|-|x||\le 2^{-l}\}}. \tag{4.5}$$

If $2^l|x| > 2$ and $2^l||t| - |x|| > 1$, then by Lemma 2, we have

$$|T_l^\delta f(x,t)| \le C\, 2^{l(n+1-n/p-N)}\, \{|x|^{-n/p-N}\, \chi_{\{|t|\le 2^{1-l}\}} + |x|^{-n/p}\, |t|^{-N} \chi_{\{|t|>2^{1-l}\}}\}. \tag{4.6}$$

Finally, when $2^l|x| \le 2$, $2^l|t| > 2$, we use Lemma 1 and thus

$$|T_l^\delta f(x,t)| \le C\, 2^{l(n-\delta)}\, |t|^{-(\delta+1)}\, \chi_{\{|x|\le 2^{1-l},\, |t|>2^{1-l}\}}. \tag{4.7}$$

Combining the estimates (4.5)–(4.7), we obtain (4.4) for $l \ge 0$, and by Lemma 6,

$$\iint_{\{|x|>2^{1-l},\, |t|>2^{1-l}\, :\, |T_l^\delta f(x,t)|>\alpha/4\}} dx\, dt \le C\, 2^{l(n+1)(p-1)}\, \alpha^{-p}.$$

Now applying Lemma 5 with $\beta p = (n+1)(1-p)$, we obtain

$$\left|\left\{(x,t) \in (I^*)^c : \sum_{l\ge 0} |T_l^\delta f(x,t)| > \alpha/4\right\}\right| \le C\, \alpha^{-p}. \tag{4.8}$$

We now fix $l < 0$. Let $P_{k,l,x}(y,s)$ denote the N-th order Taylor polynomial of the function $(y,s) \to K_{k,l}(|x-y|, t-s)$ expanded about the origin, the center of the cube. Now $P_{k,l,x} = 2^{l(n+1)}\, P_{k,0,x}(2^l\cdot, 2^l\cdot)$ for fixed k and l. Then using the moment conditions on f,

$$\begin{aligned} T_{k,l}^\delta f(x,t) &= \iint_{I^*} f(y,s)\, 2^{l(n+1)}\, K_{k,0}(2^l|x-y|, 2^l(t-s))\, dy\, ds \\ &= \iint_{I^*} f(y,s)\, 2^{l(n+1)}\, [K_{k,0}(2^l|x-y|, 2^l(t-s)) - P_{k,0,x}(2^l|y|, 2^l s)]\, dy\, ds. \end{aligned}$$

A straightforward calculation shows that the absolute value of the last term is dominated by

$$C \iint_{I^*} |f(y,s)|\, 2^{l(n+1)} \sum_{|\gamma|=N+1} |K_{k,0}^{(\gamma)}(2^l|x|, 2^l t)|\, |2^l(y,s)|^{N+1}\, dy\, ds.$$

We apply Lemma 3 and the same arguments used for (4.5)–(4.7) and thus obtain the bounds (4.4) for $l < 0$. Moreover, by Lemma 6 we see that

$$\iint_{\{|x|>2^{1-l},\, |t|>2^{1-l}\, :\, |T_l^\delta f(x,t)|>\alpha/4\}} dx\, dt \le C\, 2^{l\{(n+N+2)p-(n+1)\}}\, \alpha^{-p}.$$

Since $(n+1) < (n+N+2)p$, by Lemma 5 with $\beta p = (n+N+2)p - (n+1)$, we obtain

$$\left|\left\{(x,t) \in (I^*)^c : \sum_{l<0} |T_l^\delta f(x,t)| > \alpha/4\right\}\right| \le C\, \alpha^{-p}. \tag{4.9}$$

Putting together the estimates (4.8) and (4.9), we obtain (4.3) for the cube I.

Suppose now that f is a (p,N)-atom ($N \ge (n+1)(1/p-1)$), supported in a cube Q of diameter 2^R centered at (x_Q, t_Q). By translation invariance we can assume

$(x_Q, t_Q) = (0,0)$. Let $h(x,t) = 2^{R(n+1)/p} f(2^R x, 2^R t)$. Then h is an atom supported in the cube I centered $(0,0)$. But this implies

$$\begin{aligned} T_l^\delta f(x,t) &= \int_R \int_{R^n} 2^{-R(n+1)/p} h\left(\frac{(x-y)}{2^R}, \frac{(t-s)}{2^R}\right) G_l(y,s)\, dy\, ds \\ &= 2^{-R(n+1)/p}\, 2^{R(n+1)} \left(G_l(2^R \cdot, 2^R \cdot) * h\right)\left(\frac{x}{2^R}, \frac{t}{2^R}\right) \\ &= 2^{-R(n+1)/p} \left(G_{l+R} * h\right)\left(\frac{x}{2^R}, \frac{t}{2^R}\right). \end{aligned}$$

If we repeat the same arguments used for (4.8) and (4.9), we get (4.3). This proves Proposition 1. □

We now proceed with the proof of Theorem 1.

Proof. If $f = \sum_{i=1}^\infty \lambda_i f_i \in H^p(\mathbb{R}^{n+1})$, $T_l^\delta f$ is well defined since each $(T_l^\delta f_i)(x,t)$ is the convolution of the atom f_i with an integrable function G_l. Using the weak type (p,p) estimates of T_l^δ for each l and by Lemma 5, we obtain that $T^\delta f_i$ satisfies a uniform weak type estimate when f_i is a (p,N)-atom $(N \geq (n+1)(1/p-1))$ in Proposition 1. Since $|T^\delta f(x,t)| \leq \sum_{i=1}^\infty |\lambda_i||T^\delta f_i(x,t)|$ and $\sum_{i=1}^\infty |\lambda_i|^p < \infty$, Theorem 1 is a consequence of Lemma 4. □

Appendix

Let $0 < p < 1$. Denote the quasi-norm $(\sup_{\alpha>0} \alpha^p |\{x \in \mathbb{R}^n : |g(x)| > \alpha\}|)^{1/p}$ of g in $L^{p,\infty}$ by $||g||_{L^{p,\infty}}$. Let $T_m f = m^\vee * f$. We define the class of Fourier multipliers $\boldsymbol{\mathcal{M}}(H^p, L^{p,\infty})(\mathbb{R}^n)$ to be the set of all bounded measurable functions m so that for all $f \in C_0^\infty(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$,

$$||T_m f||_{L^{p,\infty}} \leq C ||f||_{H^p}.$$

The best constant C is the quasi-norm of the operator T_m, and we write $||m||_{\boldsymbol{\mathcal{M}}}$ for this quantity.

Lemma 7. *Suppose f_ϵ and f are measurable functions on $\mathbb{R}^n$ and $f_\epsilon \to f$ almost everywhere. Assume that $||f_\epsilon||_{L^{p,\infty}} \leq M^{1/p}$ for some $M > 0$ and for all $\epsilon > 0$. Let $\alpha > 0$ be fixed. Then $\alpha^p |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| \leq M$.*

Proof. Suppose $\alpha^p |\{x : |f(x)| > \alpha\}| > M$. Then from the right continuity of $|\{x : |f(x)| > \alpha\}|$, there exists $\beta > \alpha$ such that $\alpha^p |\{x : |f(x)| > \beta\}| > M$. Define $E = \{x : |f(x)| > \alpha\}$. We know $|E| < \infty$. Then by Egoroff's theorem, for every $\eta > 0$, there exists $F \subset E$ such that $|E \setminus F| < \eta$ and $f_\epsilon \to f$ uniformly on F. We choose ϵ small so that $|f_\epsilon(x) - f(x)| < (\beta - \alpha)/2$ for $x \in F$. Moreover, on F the inequality $|f(x)| > \beta$ implies that $|f_\epsilon(x)| \geq |f(x)| - |f_\epsilon(x) - f(x)| > \beta - (\beta - \alpha)/2 > \alpha$.

From this, $\{x : |f(x)| > \beta\} \cap F \subset \{x : |f_\epsilon(x)| > \alpha\}$ and $\alpha^p |\{x : |f(x)| > \beta\} \cap F| \leq \alpha^p |\{x : |f_\epsilon(x)| > \alpha\}|$. Since $\{x : |f(x)| > \beta\} = (\{x : |f(x)| > \beta\} \cap F) \cup (\{x : |f(x)| > \beta\} \cap (E \setminus F))$,

$$\begin{aligned} & \alpha^p |\{x : |f(x)| > \beta\}| - \alpha^p |E \setminus F| \\ \leq\ & \alpha^p |\{x : |f(x)| > \beta\} \cap F| \\ \leq\ & \alpha^p |\{x : |f_\epsilon(x)| > \alpha\}| \leq M. \end{aligned}$$

When η is sufficiently small, this is a contradiction. □

The following is based on de Leeuw's restriction theorem [2].

Theorem 2. *Let $m(\xi', \xi'')$ be contained in the class $\boldsymbol{\mathcal{M}}(H^p, L^{p,\infty})(\mathbb{R}^{k+l})$ and be continuous. Then $m_{\xi''}(\xi') \equiv m(\xi', \xi'')$ is contained in the class $\boldsymbol{\mathcal{M}}(H^p, L^{p,\infty})(\mathbb{R}^k)$ and the multiplier norm of $m_{\xi''}$ does not exceed that of m.*

Proof. Let $f_1 \in C_0^\infty(\mathbb{R}^k) \cap H^p(\mathbb{R}^k)$, $f_{2,\epsilon} \in C_0^\infty(\mathbb{R}^l)$ with $\widehat{f_{2,\epsilon}}(\xi'') = \epsilon^{l(1/p-1)}\phi(\frac{\xi''-a}{\epsilon})$, where ϕ is supported in $B(0,1)$ (the unit ball about the origin). Define $f_\epsilon(x', x'') = (f_1 \otimes f_{2,\epsilon})(x', x'')$. From this $||f_\epsilon||_{H^p} \le A_{\check{\phi}} ||f_1||_{H^p}$. Since we have

$$\begin{aligned} & T_m(f_1 \otimes f_{2,\epsilon})(x', x'') \\ = \quad & \frac{1}{(2\pi)^{k+l}} \int_{\xi''} \int_{\xi'} m(\xi', \xi'') \widehat{f}_1(\xi') \epsilon^{l(1/p-1)} \phi(\frac{\xi''-a}{\epsilon}) e^{i\langle x', \xi' \rangle + i\langle x'', \xi'' \rangle} d\xi' \, d\xi'' \\ = \quad & T_{m^\epsilon}(f_1 \otimes \epsilon^{l/p} \check{\phi})(x', \epsilon x'') e^{i\langle x'', a \rangle} \end{aligned}$$

where $m^\epsilon(\xi', \xi'') = m(\xi', \epsilon\xi'' + a)$,

$$||T_{m^\epsilon}(f_1 \otimes \check{\phi})||_{L^{p,\infty}} \le A_{\check{\phi}} ||m||_{\boldsymbol{\mathcal{M}}} ||f_1||_{H^p}.$$

Then for all $\alpha > 0$, we have

$$\alpha^p |\{(x', x'') : |T_{m^\epsilon}(f_1 \otimes \check{\phi})(x', x'')| > \alpha\}| \le A_{\check{\phi}}^p ||m||_{\boldsymbol{\mathcal{M}}}^p ||f_1||_{H^p}^p$$

and by the Lebesgue Dominated Convergence Theorem, $T_{m^\epsilon}(f_1 \otimes \check{\phi})$ converges to $(T_{m_a} f_1) \otimes \check{\phi}$ as $\epsilon \to 0$ where $m_a(\xi') = m(\xi', a)$. So from Lemma 7, we have

$$||(T_{m_a} f_1) \otimes \check{\phi}||_{L^{p,\infty}} \le A_{\check{\phi}} ||m||_{\boldsymbol{\mathcal{M}}} ||f_1||_{H^p}.$$

Hence m_a is contained in the class $\boldsymbol{\mathcal{M}}(H^p, L^{p,\infty})(\mathbb{R}^k)$ and thus

$$||m_a||_{\boldsymbol{\mathcal{M}}} \le ||m||_{\boldsymbol{\mathcal{M}}}.$$

□

Remark 1. Even if we replace the continuity assumption of m by almost everywhere conditions, Theorem 2 still holds.

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