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A SHARP SCHWARZ INEQUALITY ON THE BOUNDARY

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Abstract. A number of classical results reflect the fact that if a holomorphic function maps the unit disk into itself, taking the origin into the origin, and if some boundary point b maps to the boundary, then the map is a magnification at b. We prove a sharp quantitative version of this result which also sharpens a classical result of Loewner.

A recent paper of the author and Min Ru [OR] makes use of a new Schwarz-type lemma on surfaces (Lemma 2.1 of that paper.) An effort to understand the geometry underlying the proof of that lemma led the author to a much more general Ahlfors-Schwarz Lemma on surfaces, and to an apparently new link between lemmas of that sort and the standard comparison theorems of Riemannian geometry [O].

Looking back at the comparable situation in the classical case, one finds the following:

The standard Schwarz Lemma states that an analytic function $f(z)$ mapping the unit disk into itself, with $f(0) = 0$, must map each smaller disk $|z| < r < 1$ into itself and (as a result) satisfy $|f'(0)| \leq 1$. Furthermore, unless f is a rotation, one has strict inequality $|f'(0)| < 1$ and f maps each disk $|z| \leq r < 1$ into a strictly smaller one.

It is an elementary consequence of Schwarz' Lemma that if f extends continuously to some boundary point b with $|b| = 1$, and if $|f(b)| = 1$ and $f'(b)$ exists, then $|f'(b)| \geq 1$. It is also true that one again has strict inequality unless f is a rotation, but that does not follow from the standard Schwarz inequality; one needs a stronger form where one has a quantitative bound on how *much* each disk $|z| \leq r < 1$ is shrunk if f is not a rotation.

These facts are well known. (See, for example, Carathéodory [C2], §§296-301.) However, what does not seem to have been observed, at least in print, is that there is a sharp boundary inequality (Lemma 1 below) from which the above facts follow. Although the components of the proof can all be found in the literature[1] (in particular, the sections of Carathéodory's book cited above), it seems worth presenting the sharp inequalities of Lemmas 1 and 3 below together with direct elementary proofs.

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[1] I would like to thank the referee for providing details of this, and for other useful comments.

Lemma 1 (The boundary Schwarz Lemma)**.** *Let $f(z)$ satisfy*

(a) *$f(z)$ is analytic for $|z| < 1$,*
(b) *$|f(z)| < 1$ for $|z| < 1$,*
(c) *$f(0) = 0$,*
(d) *for some b with $|b| = 1$, $f(z)$ extends continuously to b, $|f(b)| = 1$, and $f'(b)$ exists.*

Then

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}. \tag{1}$$

Corollary 1. *Under hypotheses* (a) - (d)*,*

$$|f'(b)| \geq 1 \tag{2}$$

and

$$|f'(b)| > 1 \text{ unless } f(z) = e^{i\alpha} z, \ \alpha \text{ real.} \tag{3}$$

Proof. Inequalities (2) and (3) follow immediately from (1) together with the standard Schwarz Lemma. ☐

Corollary 2. *Let f satisfy conditions* (a), (b), (c) *of the lemma, and suppose that f extends continuously to an arc C on $|z| = 1$, with $|f(z)| = 1$ on C. Then the length s of C and the length σ of $f(C)$ satisfy*

$$\sigma \geq \frac{2}{1 + |f'(0)|} s. \tag{4}$$

Proof. By the reflection principle, f extends to be analytic on the interior of C and therefore satisfies condition (d) of Lemma 1. Hence (4) follows from (1). ☐

Remarks 1. 1. Again by the standard Schwarz Lemma, (4) implies that $\sigma \geq s$, and $\sigma > s$ unless f is a rotation. That is the content of a classical theorem of Loewner [L]. (See also Velling [V].)

2. The length σ of $f(C)$ is to be taken with multiplicity, if $f(C)$ is a multiple covering of the image.

3. Inequality (1) is sharp, with equality possible for each value of $|f'(0)|$.

4. One can drop the condition (c) that $f(0) = 0$. Analogous results hold for any value of $f(0)$. See Lemma 3, the General Boundary Lemma, below.

5. One does not need to assume that f extends continuously to b. For example, if f has a radial limit c at b, with $|c| = 1$, and if f has a radial derivative at b, then that derivative also satisfies the inequality (1). More generally, if for some b with $|b| = 1$ there exists a sequence z_n such that $z_n \to b$ and $f(z_n) \to c$ with $|c| = 1$, then

$$\underline{\lim}_{z_n \to b} \left| \frac{f(z_n) - c}{|z_n| - |b|} \right| \geq \underline{\lim}_{z_n \to b} \frac{1 - |f(z_n)|}{1 - |z_n|} \geq \frac{2}{1 + |f'(0)|}. \tag{5}$$

Both Lemma 1 and the statement about radial limits are immediate consequences, since in either case we may choose $z_n = t_n b$ for t_n real, $t_n \to 1$, and the left-hand side of (5) becomes $|f'(b)|$.

Lemma 2 (Interior Schwarz Lemma)**.** *Let $f(z)$ satisfy conditions* (a),(b),(c) *of Lemma 1. Then*

$$|f(z)| \leq |z| \frac{|z| + |f'(0)|}{1 + |f'(0)||z|} \text{ for } |z| < 1. \tag{6}$$

Proof. Let $g(z) = \frac{f(z)}{z}$. Then by the standard Schwarz Lemma, either f is a rotation, or else $|g(z)| < 1$ for $|z| < 1$. In the former case, $|f'(0)| = 1$ and (6) holds trivially. So we need only consider the second case, where $|g(z)| < 1$. Furthermore, since inequality (6) is unaffected by rotations, we may assume that $g(0) = f'(0) = a$, where $0 \leq a < 1$. Then (6) is equivalent to

$$|g(z)| \leq \frac{|z| + a}{1 + a|z|} \text{ for } |z| < 1, \text{ with } a = g(0). \tag{7}$$

But that is an immediate consequence of the standard Schwarz-Pick version of the Schwarz Lemma, which says that g must map each disk $|z| < r$ into the image of that disk under the linear fractional map

$$G(z) = \frac{z + a}{1 + az}$$

which is a circular disk whose diameter is the interval

$$\left[\frac{a - r}{1 - ar}, \frac{a + r}{1 + ar}\right]$$

of the real axis.

Hence,

$$|z| = r \Rightarrow |g(z)| \leq \frac{a + r}{1 + ar} = \frac{|z| + a}{1 + a|z|},$$

which proves (7), and hence (6). □

Remarks 2. 1. For related sharpened forms of the interior Schwarz Lemma, see Mercer [M].

2. Inequality (7) is sharp, with equality for $g(z) = G(z)$, $z = r$. Hence, inequality (6) is sharp, with equality for the function

$$f(z) = z\frac{z + a}{1 + az}, \ 0 \leq a < 1,$$

when z is on the positive real axis. The same function gives equality in (1) when $b = 1$.

3. When f is not a rotation, (6) is a strict improvement on the standard Schwarz Lemma, since the second factor on the right is strictly less than 1 when $|f'(0)| < 1$.

Proof of Lemma 1. Let f satisfy conditions (a), (b), (c) of Lemma 1. Then, using the upper bound (6) for $|f(z)|$, we have for any b and c with $|b| = 1$, $|c| = 1$,

$$\left|\frac{f(z) - c}{|z| - |b|}\right| \geq \frac{1 - |f(z)|}{1 - |z|} \geq \frac{1 + |z|}{1 + |f'(0)||z|}.$$

As $|z| \to 1$, the right-hand side tends to $\frac{2}{1+|f'(0)|}$.

This proves (5), and as noted in Remark 1.5 above, Lemma 1 follows. □

Lemma 3 (The General Boundary Lemma)**.** *Under hypotheses* (a), (b), *and* (d) *of Lemma 1, one has*

$$|f'(b)| \geq \frac{2}{1 + |F'(0)|}\frac{1 - |f(0)|}{1 + |f(0)|}, \tag{8}$$

where F is defined in (9) *below and satisfies* $|F'(0)| \leq 1$.

Proof. Let

$$F(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}. \tag{9}$$

Then F satisfies the hypotheses of Lemma 1, and therefore

$$|F'(b)| \geq \frac{2}{1 + |F'(0)|}. \tag{10}$$

But a calculation gives

$$F'(z) = f'(z)\frac{1 - |f(0)|^2}{[1 - \overline{f(0)}f(z)]^2}.$$

Since $|f(b)| = 1$ implies

$$|1 - \overline{f(0)}f(b)| \geq 1 - |\overline{f(0)}f(b)| = 1 - |f(0)|,$$

we have

$$|F'(b)| = |f'(b)|\frac{1 - |f(0)|^2}{|1 - \overline{f(0)}f(b)|^2} \leq |f'(b)|\frac{1 + |f(0)|}{1 - |f(0)|}. \tag{11}$$

Combining (10) and (11) yields (8). □

Remarks 3 (Concluding Remarks). 1. An interesting special case of Lemma 1 is when $f'(0) = 0$, in which case inequality (1) implies $|f'(b)| \geq 2$. Clearly equality holds for

$$f(z) = e^{i\alpha}z^2, \quad \alpha \text{ real.} \tag{12}$$

Furthermore, that is the only case of equality; the same type of argument used to prove Lemmas 1 and 2 yields a stronger inequality that implies $|f'(b)| > 2$ unless f is of the form (12). More generally, the argument of the standard Schwarz Lemma shows that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies (a), (b) of Lemma 1 and if

$$a_0 = a_1 = \cdots a_{k-1} = 0, \tag{13}$$

then $|a_k| \leq 1$, and $|a_k| = 1$ if and only if

$$f(z) = e^{i\alpha}z^k, \quad \alpha \text{ real.} \tag{14}$$

Furthermore, either (14) holds, or else $|f(z)| < |z|^k$ for $|z| < 1$. The argument of Lemma 2 yields the stronger result that

$$|f(z)| \leq |z|^k \frac{|z| + |a_k|}{1 + |a_k||z|}. \tag{15}$$

Using (15) in the proof of Lemma 1 then shows that if also condition (d) of Lemma 1 holds, then

$$|f'(b)| \geq k + \frac{1 - |a_k|}{1 + |a_k|}. \tag{16}$$

It follows that $|f'(b)| \geq k$, with equality only if f is of the form (14).

2. A corollary of Lemma 3 is that under the same hypotheses, one has

$$|f'(b)| \geq \frac{1 - |f(0)|}{1 + |f(0)|} \tag{17}$$

and the inequality is strict unless f is an automorphism of the unit disk. In this context, see Carathéodory [C1], pp. 54-55, on Julia's Theorem, and [C2], pp. 25-27.

3. For related results, and other types of boundary Schwarz Lemmas, see Carathéodory [C2], pp. 28-32, on the converse of Julia's Theorem, Pommerenke [P], p. 71, on the Julia-Wolff Lemma, and the paper of Burns and Krantz [BK]. See also [PS], paragraph **291** on p. 162 and p. 373, which gives the weaker inequality (2) together with the additional observation that if $b = f(b) = 1$, then $f'(1)$ is real and positive. For a different view of this last fact in the context of angular derivatives, see section 299 of Carathéodory [C2].

References

[BK] Burns, D.M. and Krantz, S.G. "Rigidity of holomorphic mappings and a new Schwarz Lemma at the boundary." *J. Amer. Math. Soc.* **7** (1994) 661-676. MR **94j:**32016

[C1] Carathéodory, C. *Conformal Representation* Cambridge University Press 1952. MR **13:**734a

[C2] Carathéodory, C. *Theory of Functions* Vol. 2, New York, Chelsea 1954. MR **15:**612b

[L] Löwner. K. "Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I." *Math. Ann.* **89** (1923) 103-121.

[M] Mercer, P.R. "Sharpened versions of the Schwarz Lemma." *J. Math. Anal. Appl.* **205** (1997) 508-511. MR **98e:**30016

[O] Osserman, R. "A new variant of the Schwarz-Pick-Ahlfors Lemma." *Manuscripta Mathematica* **100** (1999) 123–124.

[OR] Osserman, R. and Ru, M. "An estimate for the Gauss curvature of minimal surfaces in R^m whose Gauss map omits a set of hyperplanes." *J. Differential Geometry* **46** (1997) 578-593. MR **98k:**53014

[PS] Pólya, G. and Szegö, G. *Problems and Theorems of Analysis* Vol. 2, New York, Springer-Verlag 1972. MR **57:**5529

[P] Pommerenke, Ch. *Boundary Behaviour of Conformal Maps*, New York, Springer-Verlag 1992. MR **95b:**30008

[V] Velling, J.A. "The uniformizations of rectangles, an exercise in Schwarz's Lemma." *Amer. Math. Monthly* **39** (1992) 112-115. MR **93b:**30008

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