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# STRONGLY ASYMPTOTICALLY STABLE FROBENIUS-PERRON OPERATORS

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*Dedicated to Professor Alexandra Bellow in celebration of her achievements in all the aspects of being that involve mathematics*

Abstract. Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space and let $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ be a Frobenius-Perron operator.

In 1997 Bartoszek and Brown proved that if $T$ overlaps supports and if there exists $h \in L^1(X, \Sigma, \mu)$, $h > 0$ on $X$, such that $Th = h$, then $T$ is (strongly) asymptotically stable.

In the note we prove that instead of assuming that $h > 0$ on $X$, it is enough to assume that $h \geq 0$ and $h \neq 0$. More precisely, we prove that $T$ is asymptotically stable if and only if $T$ overlaps supports and there exists $h \in L^1(X, \Sigma, \mu)$, $h \geq 0$, $h \neq 0$, such that $Th = h$.

## 1. Introduction

Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space (throughout the paper, $(X, \Sigma, \mu)$ denotes a given $\sigma$-finite measure space). A measurable transformation $\phi : X \to X$ is called nonsingular if $\mu(\phi^{-1}(A)) = 0$ whenever $\mu(A) = 0$. A positive contraction $S : L^\infty(X, \Sigma, \mu) \to L^\infty(X, \Sigma, \mu)$ is called a Koopman operator if there exists a nonsingular transformation $\phi : X \to X$ such that $Su = u \circ \phi$ for every $u \in L^\infty(X, \Sigma, \mu)$. Using the Radon-Nikodym theorem it can be shown (see [5]) that every Koopman operator has a predual; that is, given a Koopman operator $S : L^\infty(X, \Sigma, \mu) \to L^\infty(X, \Sigma, \mu)$, there exists a positive contraction $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ such that $T' = S$, where $T'$ is the dual of $T$; the operator $T$ is called a Frobenius-Perron operator (for a detailed discussion of the Frobenius-Perron operators see the book by Lasota and Mackey [5]).

Let $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ be a positive contraction.

We say that $T$ overlaps supports if for every $f, g \in L^1(X, \Sigma, \mu)$, $f \geq 0$, $f \neq 0$, $g \geq 0$, $g \neq 0$, there exists $n \in \mathbb{N}$ ($n$ depends on $f$ and $g$) such that $T^n f \wedge T^n g \neq 0$.

If $f \in L^1(X, \Sigma, \mu)$, $f \geq 0$, $||f|| = 1$, then it is often the custom to call $f$ a density. The operator $T$ is called (strongly) asymptotically stable if there exists a unique density $f$ such that the sequence $(T^n g)_{n \in \mathbb{N}}$ converges in the norm topology of $L^1(X, \Sigma, \mu)$ to $f$ for every density $g$ in $L^1(X, \Sigma, \mu)$.

If $g \in L^1(X, \Sigma, \mu)$, we say that $g$ is $T$-invariant (or, simply, invariant) if $Tg = g$.

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In 1997 Bartoszek and Brown [2] proved that if $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is a Frobenius-Perron operator which overlaps supports and has an invariant density $f$ such that $f > 0$ on $X$, then $T$ is asymptotically stable. Our goal in this note is to prove that a Frobenius-Perron operator is asymptotically stable if and only if it overlaps supports and has an invariant density.

It is of interest to point out that there exist Frobenius-Perron operators to which the Bartoszek-Brown result [2] does not apply. More precisely, it may happen that an asymptotically stable Frobenius-Perron operator has a (necessarily unique) invariant density $f$ and the set $\{f = 0\}$ has strictly positive measure, as we can see from the following examples.

**Example 1.1.** The following example seems to be the simplest one: let $X = \{1, 2\}$, let $\Sigma$ be the collection of all the subsets of $X$, let $\mu$ be the counting measure defined on $(X, \Sigma)$, and let $\phi : X \to X$ be defined by $\phi(1) = \phi(2) = 1$. Then the Frobenius-Perron operator $T$ generated by $\phi$ is strongly asymptotically stable and has the invariant density $f \in L^1(X, \Sigma, \mu)$ defined by $f(1) = 1$ and $f(2) = 0$.

**Example 1.2.** Let $\phi : \mathbb{N} \to \mathbb{N}$ be defined by $\phi(1) = \phi(2) = 1$ and $\phi(n) = n - 1$ for every $n \geq 3$. If we consider the counting measure on $\mathbb{N}$, then obviously $\phi$ is nonsingular. The corresponding Frobenius-Perron operator $T : l^1 \to l^1$ is defined by

$$T(a_1, a_2, a_3, a_4, a_5, \ldots) = (a_1 + a_2, a_3, a_4, a_5, \ldots)$$

for every $(a_n)_{n \in \mathbb{N}} \in l^1$. Clearly, $T$ is strongly asymptotically stable and $(1, 0, 0, 0, \ldots)$ is the invariant density of $T$.

**Example 1.3.** In the previous two examples the $L^1$-spaces were generated by counting measures. Let us now consider an example that involves a continuous measure. To this end, let $X = [0, 2]$, let $\Sigma$ be the $\sigma$-algebra of all Lebesgue measurable subsets of $[0, 2]$, let $\lambda$ be the Lebesgue measure on $[0, 2]$, and let $\phi : [0, 2] \to [0, 2]$ be defined by $\phi(x) = 2x \bmod 1$ for $x \in [0, 1]$ and $\phi(x) = x - 1$ for $x \in (1, 2]$. Using the example discussed on p. 116 of Horowitz [3], we conclude that if we let $T$ be the Frobenius-Perron operator corresponding to $\phi$, then $T$ is strongly asymptotically stable and $1_{[0,1]}$ is the invariant density of $T$.

Bartoszek and Brown proved (see Proposition 1 of [2]) that if $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is a positive contraction which overlaps supports, and if there exists a $T$-invariant density $g$ such that $g > 0$ on $X$, then the sequence $(T^n)_{n \in \mathbb{N}}$ converges in the weak operator topology to a one-dimensional projection; however, the example discussed on pp. 362-364 of Akcoglu and Boivin [1] shows that, in general, $T$ is not strongly asymptotically stable. Thus, in Theorem 1 of [2] and in the main result of this note, we cannot assume that $T$ is merely a positive contraction rather than a Frobenius-Perron operator.

The paper is organized as follows: in the next section (Section 2) we discuss several lemmas which are needed in the last section (Section 3), where we prove the main result of this note.

## 2. Invariant densities and support overlapping

Let $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ be a positive contraction.

Given $A \in \Sigma$, set $\Sigma_A = \{B \in \Sigma \mid B \subseteq A\}$. Clearly, $\Sigma_A$ is a $\sigma$-algebra on $A$ and the function $\mu_A : \Sigma_A \to \mathbb{R} \cup \{+\infty\}$, $\mu_A(B) = \mu(B)$ for every $B \in \Sigma_A$

is a measure on $(A, \Sigma_A)$. Set also $L^1(A) = \{g \in L^1(X, \Sigma, \mu) \mid g1_A = g\}$ and $L^\infty(A) = \{u \in L^\infty(X, \Sigma, \mu) \mid u1_A = u\}$.

A measurable subset $A$ of $X$ is called $T$-absorbing if $Tg \in L^1(A)$ whenever $g \in L^1(A)$. If $A$ is $T$-absorbing, we define an operator $T_{(A)} : L^1(A, \Sigma_A, \mu_A) \to L^1(A, \Sigma_A, \mu_A)$ as follows: if $g \in L^1(A, \Sigma_A, \mu_A)$, then let $\tilde{g} \in L^1(X, \Sigma, \mu)$ be such that $\tilde{g} = g$ on $A$ and $\tilde{g} = 0$ on $X \backslash A$; now set $T_{(A)}g = (T\tilde{g})_{|A}$, where $(T\tilde{g})_{|A}$ is the restriction of $T\tilde{g}$ to $A$.

**Lemma 2.1.** *Let $\psi : X \to X$ be a nonsingular transformation, and let*

$$T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$$

*be the Frobenius-Perron operator corresponding to $\psi$ and to $\mu$. If $A \in \Sigma$, $\mu(A) > 0$ is a $T$-absorbing set, then $T_{(A)}$ is also a Frobenius-Perron operator.*

The proof of the above lemma follows from Theorem 1.9, p. 119 of Krengel's book [4]. Indeed, since $T'g = g \circ \psi$ whenever $g \in L^\infty(X, \Sigma, \mu)$, it follows that $A$ is absorbing if and only if $\psi(A) \subseteq A$ $\mu$ - a.e. Thus, (by possibly redefining $\psi$ on a $\mu$-negligible set) $T_{(A)}$ is a Frobenius-Perron operator corresponding to the restriction of $\psi$ to $A$.

Given a positive contraction $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ we will consider the Hopf decomposition (see Section 3.1 of [4]) of $X$ into the conservative part $C$ and the dissipative part $D = X \backslash C$ generated by $T$. We will also use the fact (see the proof of Corollary 1 of [7]) that if $T$ overlaps supports and has an invariant density, then the invariant density is unique (that is, if $T$ overlaps supports, and $f$, $g$ are densities such that $Tf = f$ and $Tg = g$, then $f = g$).

**Lemma 2.2.** *Assume that $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is a Frobenius-Perron operator which overlaps supports and has a (necessarily) unique invariant density $f$. Then $C = \{f > 0\}$ and $\inf_{n \in \mathbb{N}} T'^n 1_D = 0$, where $T'$ is the dual of $T$.*

*Proof.* It is well known that $\{f > 0\} \subseteq C$. If we assume that $\{f > 0\} \neq C$, then using the observations made on p. 126 of Krengel's book [4], and taking into consideration that $\{f > 0\}$ is a $T$-absorbing set, we obtain that $C \backslash \{f > 0\}$ is also $T$-absorbing. We obtain a contradiction since on one hand $\{f > 0\}$ and $C \backslash \{f > 0\}$ are $T$-absorbing sets, while on the other hand $T$ overlaps supports.

Clearly, in order to complete the proof of the lemma, it is enough to prove that $\sup_n T'^n 1_C = 1_X$. To this end, note that since $T'$ is a Koopman operator, it follows that $\sup_n T'^n 1_C = 1_B$ for some $B \in \Sigma$. Assume that $B \neq X$; that is, assume that $\mu(X \backslash B) > 0$. Then there exists $g \in L^1(X, \Sigma, \mu)$, $g \geq 0$, $g \neq 0$, $g1_B = 0$. Since

$$\begin{aligned} \int (T^n f) \wedge (T^n g) d\mu &= \int (f \wedge (T^n g)) 1_C d\mu \leq \int (T^n g) 1_C d\mu \\ &\leq \int g(T'^n 1_C) d\mu \leq \int g 1_B d\mu = 0 \end{aligned}$$

for every $n \in \mathbb{N}$, we obtain a contradiction since $T$ overlaps supports. Q.E.D.

## 3. Asymptotic stability

As mentioned in the Introduction, our goal in this section is to prove the main result of this note (Corollary 3.2 below).

**Theorem 3.1.** *Let $T$ be a positive contraction of $L^1(X,\Sigma,\mu)$ with dual $T'$ and with conservative and dissipative parts $C$ and $D$, respectively. The following assertions are equivalent:*

(a) *$T$ is strongly asymptotically stable.*
(b) *$\inf\limits_{n\in\mathbb{N}} T'^n 1_D = 0$ and $T_{(C)}$ is strongly asymptotically stable.*

*Proof.* (a) ⇒ (b). Using a result of Helmberg (see Theorem 3.3, p. 175 of [4]), we obtain that $\inf\limits_n T'^n 1_D = 0$.

Clearly, the fact that $T$ is strongly asymptotically stable implies that $T_{(C)}$ is also strongly asymptotically stable.

(b) ⇒ (a). Since $T_{(C)}$ is strongly asymptotically stable, there exists $h \in L^1(C,\Sigma_C,\mu_C)$, $h \geq 0$, $||h|| = 1$ such that $T_{(C)}h = h$. Let $f \in L^1(X,\Sigma,\mu)$ be defined as follows: $f = h$ on $C$ and $f = 0$ on $D$.

In order to complete the proof of the theorem, it is enough to prove that $(T^n g)_{n\in\mathbb{N}}$ converges to $f$ in the norm topology of $L^1(X,\Sigma,\mu)$ whenever $g$ is a density in $L^1(X,\Sigma,\mu)$. Since every $u \in L^1(X,\Sigma,\mu)$ can be written in the form $u = u1_C + u1_D$, and since the strong asymptotic stability of $T_{(C)}$ implies that $(T^n(u1_C))_{n\in\mathbb{N}}$ converges in norm to $f \cdot \int u1_C d\mu$, it follows that it is actually enough to prove that $(T^n g)_n$ converges to $f$ whenever $g$ is a density in $L^1(X,\Sigma,\mu)$ such that $g1_D = g$.

To this end, set $T_D = 1_D T 1_D$ and note that $D$ is a strong zero set in the terminology of [8] (that is, the sequence of operators $(T_D^n)_{n\in\mathbb{N}}$ converges strongly to zero). Indeed,

$$\int T_D^n u d\mu = \int u((1_D T' 1_D)^n 1_X) d\mu \leq \int u(T'^n 1_D) d\mu$$

for every $n \in \mathbb{N}$ and $u \in L^1(X,\Sigma,\mu)$, $u \geq 0$. Since $(T'^n 1_D)_n$ is a decreasing sequence (of elements of $L^\infty(X,\Sigma,\mu)$), and since $\inf\limits_n T'^n 1_D = 0$, it follows that $\lim\limits_{n\to\infty} ||T_D^n u|| = 0$ for every $u \in L^1(X,\Sigma,\mu)$.

Now, let $g$ be a density in $L^1(X,\Sigma,\mu)$ such that $g1_D = g$, and let $\epsilon \in \mathbb{R}$, $\epsilon > 0$. Since $D$ is a strong zero set, it follows that there exists $n_1 \in \mathbb{N}$ such that $||T_D^n g|| < \frac{\epsilon}{3}$ for every $n \geq n_1$.

By Lemma 2.2 of Lin [6] (see also Lemma 2.1 of [8]), it follows that there exist $g_1, g_2, g_3, \ldots, g_{n_1} \in L^1(X,\Sigma,\mu)$, $g_i \geq 0$, $g_i 1_C = g_i$ for every $i = 1,2,3,\ldots,n_1$ such that $T^{n_1} g = T_D^{n_1} g + \sum\limits_{i=1}^{n_1} T^{n_1 - i} g_i$.

Since $g_i 1_C = g_i$, it follows that $\lim\limits_{n\to\infty} \left|\left| T^n g_i - \left(\int g_i d\mu\right) f \right|\right| = 0$ for every $i = 1,2,\ldots,n_1$. Accordingly, there exists $n_2 \in \mathbb{N}$ such that $\left|\left| T^n g_i - \left(\int g_i d\mu\right) f \right|\right| \leq \frac{\epsilon}{3}||g_i||$ for every $n \geq n_2$ and $i = 1,2,\ldots,n_1$.

Let $n_\epsilon = n_1 + n_2$.

Since

$$\sum_{i=1}^{n_1} ||g_i|| = \left|\left|\sum_{i=1}^{n_1} g_i\right|\right| = ||T^{n_1} g - T_D^{n_1} g|| = ||T^{n_1} g|| - ||T_D^{n_1} g|| > 1 - \frac{\epsilon}{3},$$

it follows that

$$\begin{aligned}||T^n g - f|| &= ||T^{n-n_1}(T_D^{n_1} g + \sum_{k=1}^{n_1} T^{n_1-k} g_k) - f|| \\ &\leq \left|\left|T^{n-n_1} T_D^{n_1} g - \left(1 - \sum_{i=1}^{n_1} ||g_i||\right) f\right|\right| \\ &+ \sum_{k=1}^{n_1} \left|\left|T^{n-k} g_k - \left(\int g_k d\mu\right) f\right|\right| \\ &\leq ||T_D^{n_1} g|| + \left(1 - \sum_{i=1}^{n_1} ||g_i||\right) + \frac{\epsilon}{3} \sum_{i=1}^{n_1} ||g_i|| \\ &< \frac{\epsilon}{3} + \left(1 - \left(1 - \frac{\epsilon}{3}\right)\right) + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

for every $n \geq n_\epsilon$. Q.E.D.

**Corollary 3.2.** *Let $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ be a Frobenius-Perron operator. The following assertions are equivalent:*

(a) *$T$ is strongly asymptotically stable.*
(b) *$T$ has an invariant density and overlaps supports.*

*Proof.* (a) ⇒ (b). If $g$ is a density, then the sequence $(T^n g)_{n \in \mathbb{N}}$ converges in the norm topology of $L^1(X, \Sigma, \mu)$ to a density $f$ which is $T$-invariant.

Let $g$ and $h$ be densities in $L^1(X, \Sigma, \mu)$. Since $\lim\limits_{n \to \infty} ||T^n g - T^n h|| = 0$, and since

$$||T^m g - T^m h|| = ||T^m g + T^m h - 2((T^m g) \wedge (T^m h))||$$

for every $m \in \mathbb{N}$, it follows that there exists $n \in \mathbb{N}$ such that $(T^n g) \wedge (T^n h) \neq 0$. Thus, $T$ overlaps supports.

(b) ⇒ (a). Let the conservative and the dissipative parts of $X$ generated by $T$ be $C$ and $D$, respectively.

Since $T$ has an invariant density, it follows that $C \neq \varnothing$.

Since $C$ is a $T$-absorbing set, Lemma 2.1 tells us that $T_{(C)}$ is a Frobenius-Perron operator, while Lemma 2.2 implies that $T_{(C)}$ satisfies the conditions of Theorem 1 of [2]; hence $T_{(C)}$ is strongly asymptotically stable.

Finally, using Lemma 2.2 and Theorem 3.1 we conclude that $T$ is strongly asymptotically stable. Q.E.D.

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